Excursion Representation of Stochastic Block Model

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joint work with David Clancy and Vlada Limic

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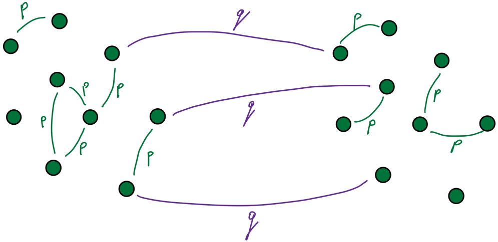
Stochastic Block Model

Stochastic Block Model $G(n, p, q)$ **is a random graph such that:**

- consists of nm vertices divided into m subsets $(m = 2)$;
- edges are drown independently;
- intra class edges appear with probability $p = p_n$;
- inter class edges appear with probability $q = q_n$.

SBM $n = 9$ $m = 2$



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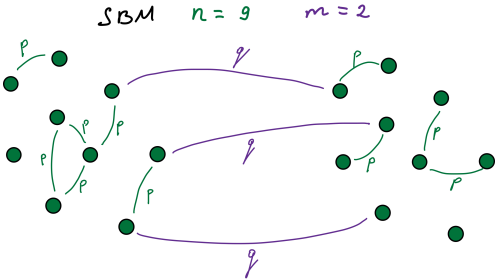

We are interested in the scaling limit as $n \to \infty$ and $p_n, q_n \to 0$.

Largest Connected Component of SBM

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It is well-known:

- If $p_n = q_n = \frac{a}{mn}$, then SBM is an Erdős-Rényi graph for which:
  - for $a > 1$, $C_1(n) \sim \Theta(n)$;
  - for $a < 1$, $C_1(n) \sim \Theta(\ln n)$; (Erdős, Rényi '60, '61)
  - for $a = 1$, $C_1(n) \sim \Theta(n^{2/3})$.

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- If $p_n = \frac{a}{mn}$, $q_n = \frac{b}{mn}$, then
  - $a + (m-1)b > m$, $C_1(n) \sim \Theta(n)$;
  - $a + (m-1)b \leq m$, $C_1(n) \sim o(n)$. (Bollobás, Janson, Riordan '07)

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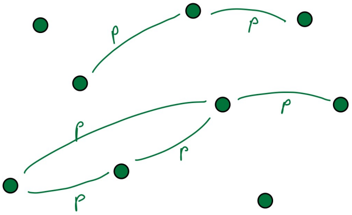
We are interested in the new critical regime: $q_n \ll p_n \sim \frac{1}{n}$.

Scaling Limit of Erdős-Rényi Graphs

$G(n,p)$ – an Erdős-Rényi random graph with n vertices and edges appearing with prob.

$$p = p_n(t) = \frac{1}{n} + \frac{t}{n^{4/3}}, \quad t \in \mathbb{R}$$



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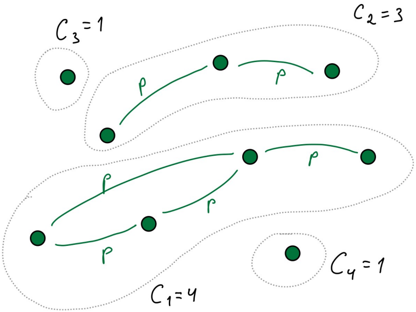
Define

$$X^{(n)}(t) := \frac{1}{n^{2/3}}(C_1, C_2, \dots, C_k, 0, 0, \dots\dots),$$

where $C_k = C_k(n, t)$ is the size of the k-th largest connected component.



Scaling Limit of Erdős-Rényi Graphs

Theorem. **(Aldous '97, Anmerdariz '01, Limic '98,'19)**

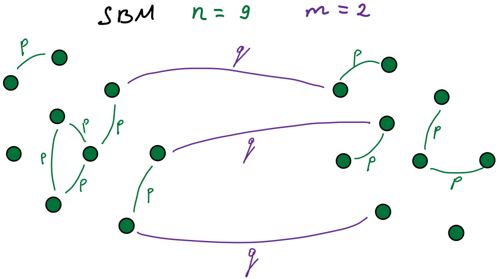
For every $t \in \mathbb{R}$ the sequence $X^{(n)}(t)$ converges in l^2 in distribution to a **standard Multiplicative Coalescent** (MC) $X^*(t)$, where

$X^*(t)$ is the ordered excursion lengths of the Brownian motion with parabolic drift

$$B^t(r) := B(r) - \frac{1}{2}r^2 + tr, \quad r \geq 0,$$

above past minima.

Stochastic Block Model



$$p = p_n(t) = \frac{1}{n} + \frac{t}{n^{4/3}} \quad q = q_n(s) = \frac{s}{n^{4/3}}, \quad t \in \mathbb{R}, \quad s \geq 0.$$

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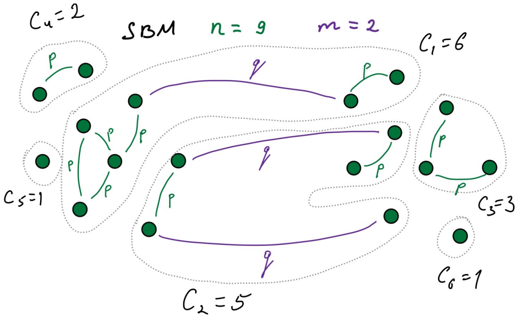

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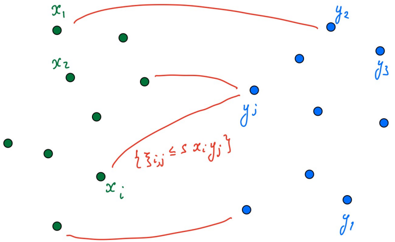
Define

$$Z^{(n)}(t,s) := \frac{1}{n^{2/3}}(C_1, C_2, \dots, C_k, 0, 0), \quad t \in \mathbb{R}, \quad s \geq 0,$$

where $C_k = C_k(n,t,s)$ is the size of the k-th largest connected component of the SBM $G(n, p_n, q_n)$

Restricted Multiplicative Merging



Let $l^2_\downarrow = \{x = (x_i)_{i\geq 1} \in l^2 : \; x_1 \geq x_2 \geq \cdots \geq 0\}$.

For $s \geq 0$ and a fixed family of indep. r.v. $\xi_{i,j} \sim \mathrm{Exp}(\mathit{rate}\ 1)$, $i, j \geq 1$, define a random map $\mathrm{RMM}_s : l^2_\downarrow \times l^2_\downarrow \to l^2_\downarrow$:

- consider coord. of $x, y \in l^2_\downarrow$ as a masses of corresponding vertices of a graph;
- for every $i, j \geq 1$ draw an edge between x_i and y_j iff $\xi_{i,j} \leq s x_i y_j$;
- define $\mathrm{RMM}_s(x, y)$ as the vector of the ordered masses of connected components.

Scaling Limit of SBM

Recall

$$p = p_n(t) = \frac{1}{n} + \frac{t}{n^{4/3}} \quad q = q_n(s) = \frac{s}{n^{4/3}}, \quad t \in \mathbb{R}, \quad s \geq 0.$$

$$Z^{(n)}(t,s) := \frac{1}{n^{2/3}}(C_1, C_2, \ldots, C_k, 0, 0), \quad t \in \mathbb{R}, \quad s \geq 0,$$

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Theorem. **(K., Limic '21)**

For every $t \in \mathbb{R}$ and $s \geq 0$ the process $Z^{(n)}(t,s)$ converges in l^2 in distribution to $\mathrm{RMM}_s(X^*(t), Y^*(t))$, where X^*, Y^* are independent standard multiplicative coalescents that are independent of ξ.

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We will call $\mathrm{RMM}_s(X^*(t), Y^*(t))$ an **Interacting Multiplicative Coalescent**

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Main Problem

Question: Does the scaling limit of the SBM admit an excursion description?

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Naive Guess: Probably, interacting MC can be described via “excursions” of a random field or a family of Brownian motions with interactions.

Standard MC as Jumps of Hitting Times

$X^*(t)$ is the ordered excursion lengths of the Brownian motion with parabolic drift

$$B^t(r) := B(r) - \frac{1}{2}r^2 + tr, \quad r \geq 0,$$

above past minima.

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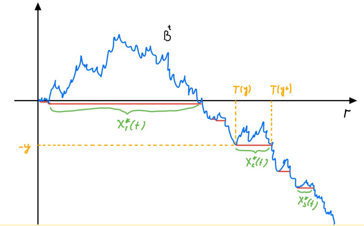

Define for $y \geq 0$

$$T(y) := \min\{r : \ B^t(r) = -y\} = \min\{r : \ \underline{B}^t(r) = -y\}$$

where $\underline{B}^t(r) = \min_{[0,r]} B^t$.

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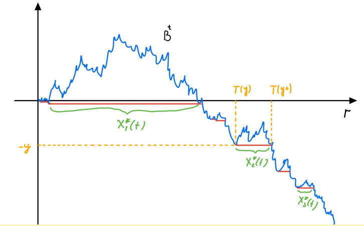

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Observation

$X^*(t)$ is the collection of decreasingly ordered jumps $T(y+) - T(y)$, $y \geq 0$.

Hitting times for fields

Consider $\vec{X} : [0, \infty)^m \to \mathbb{R}^m$ defined by

$$X_i(r_1, \ldots, r_m) = X_{i,i}(r_i) + \sum_{j \neq i} X_{i,j}(r_j),$$

such that

1. $X_{i,i}$ are continuous
2. $X_{i,j}$, $i \neq j$, are non-decreasing and continuous

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Lemma (Chaumont, Marolleau '20)

For every $\vec{y} \in [0, \infty)^m$ there exists a (component-wise) minimal solution $\vec{T} = \vec{T}(y) \in [0, \infty]^m$ to the equation

$$X_i(\vec{T}) = -y_i, \quad \forall i \text{ such that } T_i < \infty,$$

denoted by

$$\vec{T}(y) := \min\left\{\vec{r} : \ \vec{X}(\vec{r}) = -\vec{y}\right\}.$$

Scaling limit of SBM as Jumps of Hitting Times

For fixed $t \in \mathbb{R}$ and $s \geq 0$ define

$$X_{i,i}(r) := B_i^t(r) = B_i(r) - \frac{1}{2}r^2 + tr, \quad r \geq 0$$

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Let $m = 2$ and

$$X_1(r_1, r_2) = X_{1,1}(r_1) + X_{1,2}(r_2) = B_1^t(r_1) + sr_2,$$

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Theorem. (Clancy, K., Limic '23)

For every $t \in \mathbb{R}$ and $s \geq 0$, the distribution of $\mathrm{RMM}_s(X^*(t), Y^*(t))$ coincides with the law of decreasingly ordered sequence of norms of jumps $\|\vec{T}(y+) - \vec{T}(y)\|_1$, where $\|\vec{r}\|_1 = r_1 + r_2$, $r_i \geq 0$.

Construction of a continuous Curve

Note that

$$\vec{T}(y) = \min\{(r_1,r_2):\ X_1(r_1,r_2) = -y,\ \ X_2(r_1,r_2) = -y\}$$
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where $\underline{X}_1(r_1,r_2) = \underline{B}_1^t(r_1) + sr_2$, $\underline{X}_2(r_1,r_2) = sr_1 + \underline{B}_2^t(r_2)$, $\underline{B}_i^t(r) = \min_{[0,r]} B_i^t$.

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We define a curve $\gamma : [0, \infty) \to [0, \infty)^2$ by

$$\underline{X}_1(\gamma(u)) = \underline{X}_2(\gamma(u)), \quad \|\gamma(u)\|_1 = u$$

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Lemma

Take $g_i(r) := sr - \underline{B}_i^t(r)$, $\kappa := \left(g_1^{-1} + g_2^{-1}\right)^{-1}$. Then γ is uniquely determined by $\gamma_i(u) = g_i^{-1} \circ \kappa(u)$. Moreover, for every $y \geq 0$ the hitting time

$$S(y) := \inf\{u : \; X_i \circ \gamma(u) = -y\}, \quad i = 1,2$$

satisfies $\vec{T} = \gamma \circ S$, $\|\vec{T}\|_1 = \|\gamma \circ S\|_1 = S$

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satisfies $\vec{T} = \gamma \circ S$, $\|\vec{T}\|_1 = \|\gamma \circ S\|_1 = S$

$$\Longrightarrow \quad \|\vec{T}(y+) - \vec{T}(y)\|_1 = S(y+) - S(y).$$

Excursion Description of SBM along the Curve

Theorem. **(Clancy, K., Limic '23)**

Let

$$g_i(r) = sr - \underline{B}_i^t(r) = sr - \min_{[0,r]} B_i^t,$$
$$\kappa = \left(g_1^{-1} + g_2^{-1}\right)^{-1}$$
$$\gamma_i = g_i^{-1} \circ \kappa.$$

Then for every $t \in \mathbb{R}$ and $s \geq 0$, the distribution of the scaling limit of the stochastic block model $\mathrm{RMM}_s(X^*(t), Y^*(t))$ coincides with the law of ordered excursion lengths of $X_i \circ \vec{\gamma}$ above past minima, where

$$X_1(\vec{r}) = B_1^t(r_1) + sr_2, \quad r_i \geq 0$$

References

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Electronic Journal of Probability. Vol. 26 (2021), 23 pp.

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Excursion representation of Stochastic Block Model.
In preparation (2023)

Thank you!