A Quantitative Central Limit Theorem for Simple Symmetric Exclusion Process

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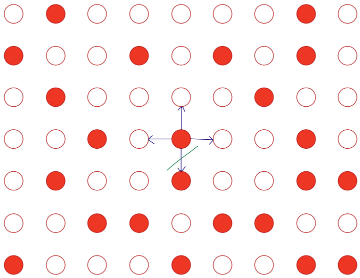
Simple symmetric exclusion process

On the d-dim discrete torus

$$\mathbb{T}_n^d := \left\{ \frac{k}{2n+1} : \ k \in \mathbb{Z}_n^d := \{-n, \dots, n\}^d \right\} \subset \mathbb{T}^d = (\mathbb{R}/\{\mathbb{Z} - 1/2\})^d$$

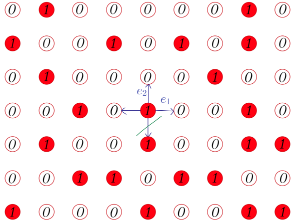
we consider a **Simple Symmetric Exclusion Process** (SSEP)



State space and generator

Particle configuration $\eta \in \{0,1\}^{\mathbb{T}_n^d}$:

$$\eta(x) = 0 \Leftrightarrow \text{side } x \text{ is empty}$$
$$\eta(x) = 1 \Leftrightarrow \text{side } x \text{ is occupied}$$



$$\eta^{x\leftrightarrow y}(z) = \begin{cases} \eta(z), & z \neq x, y, \\ \eta(y), & z = x, \\ \eta(x), & z = y, \end{cases}$$

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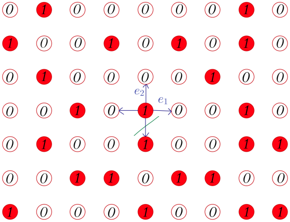

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$$\mathcal{G}_n^{EP} F(\eta) := \frac{(2n+1)^2}{2} \sum_{j=1}^{d} \sum_{x \in \mathbb{T}_n} \left[F(\eta^{x\leftrightarrow x+e_j}) - F(\eta) \right]$$

SSEP is already parabolically rescaled: space $\sim \frac{1}{2n+1}$ time $\sim (n+1)^2$!

Non-equilibrium SSEP

Let η^n_t, $t \geq 0$, be a SSEP.

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Equilibrium SSEP:

For fixed $\rho_0 \in [0,1]$, the measure $B(\rho_0)^{\otimes \mathbb{T}_n^d}$ is invariant for SSEP η_t^n:

If $\eta_0^n(x) \sim B(\rho_0)$, $x \in \mathbb{T}_n^d$, are independent then $\eta_t^n(x) \sim B(\rho_0)$, $x \in \mathbb{T}_n^d$, are independent.

$\rightsquigarrow \mathbb{E}\eta_t^n(x) = \rho_0, \ \mathrm{Var}\,\eta_t^n = \rho_0(1-\rho_0)$.

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What is $\mathbb{E}\eta_t^n(x)$?

Mean behavior

Set $\rho_t^n(x) := \mathbb{E}\eta_t^n(x)$.

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$$d\rho_t^n(x) = \mathbb{E}\mathcal{G}_n^{EP}\eta_t^n(x)dt = \mathbb{E}\frac{(2n+1)^2}{2}\sum_{j=1}^{d}\sum_{y\in\mathbb{T}_n}\left[\eta^{y\leftrightarrow y+e_j}(x) - \eta(x)\right] dt$$

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$$= \frac{(2n+1)^2}{2}\sum_{j=1}^{d}\mathbb{E}\left(\eta_t^n(x+e_j) + \eta_t^n(x-e_j) - 2\eta(x)\right)dt$$

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In particular, the empirical distribution

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$$d\langle\varphi,\tilde{\rho}_t^n\rangle = \frac{1}{2}\langle\Delta_n\varphi,\tilde{\rho}_t^n\rangle dt \quad \overset{n\to\infty}{\to} \quad d\rho_t^\infty = \frac{1}{2}\Delta\rho_t^\infty dt, \quad \rho_0^\infty = \rho_0$$

Law of large numbers

Theorem [see e.g. in Kipnis, Landim '99]

Let $\rho_0 : \mathbb{T}^d \to [0,1]$ be an initial density profile and $\eta_0^n(x) \sim B(\rho_0(x))$ be independent. Then

$$\tilde{\eta}_t^n := \frac{1}{(2n+1)^d} \sum_{x \in \mathbb{T}_n^d} \eta_t(x)\delta_x$$

converges in probability to $\rho_t^\infty(x)dx$, where $\rho_t^\infty := P_t^{HE}\rho_0$ solves

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$$= \frac{1}{2} f'(\langle\varphi, \tilde{\eta}\rangle)\langle\Delta_n\varphi, \tilde{\eta}\rangle + \frac{1}{4(2n+1)^d} f''(\langle\varphi, \tilde{\eta}\rangle)\left\langle \tau_j\tilde{\eta} + \tilde{\eta} - 2\widetilde{\eta\tau_j\eta}, \left|\partial_{n,j}\varphi\right|^2 \right\rangle + \dots,$$

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where $\tau_j \eta(x) := \eta(x + e_j)$. $\rightsquigarrow$ $d\langle \varphi, \tilde{\eta}_t^n \rangle = \langle \Delta_n \varphi, \tilde{\eta}_t^n \rangle dt + \text{mart.}$

$d\langle \text{mart.} \rangle_t = \frac{1}{(2n+1)^d} \langle \tau_j \tilde{\eta}_t^n + \tilde{\eta}_t^n - 2\widetilde{\eta_t^n \tau_j \eta_t^n}, \left| \partial_{n,j} \varphi \right|^2 \rangle dt$

Density fluctuation field and CLT

We now consider the fluctuations of the SSEP around its mean:

$$\zeta_t^n(x) := (2n+1)^{d/2} \left(\eta_t^n(x) - \rho_t^n(x)\right).$$

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can be expanded as follows

$$\begin{aligned}\mathcal{G}_n^{FF} f\left(\langle\varphi,\tilde{\zeta}\rangle\right) &= \mathcal{G}_n^{EP} f\left((2n+1)^{d/2}\langle\varphi,\tilde{\eta}-\tilde{\rho}\rangle\right)\\ &= \frac{1}{2}f'\left(\langle\varphi,\tilde{\zeta}\rangle\right)\langle\Delta_n\varphi,\tilde{\zeta}\rangle + \frac{(2n+1)^d}{4(2n+1)^d}f''\left(\langle\varphi,\tilde{\zeta}\rangle\right)\left\langle\tau_j\tilde{\eta}+\tilde{\eta}-2\widetilde{\eta\tau_j\eta},|\partial_{n,j}\varphi|^2\right\rangle\\ &+ O\left(1/n^{\frac{d}{2}+1}\right)\end{aligned}$$

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Again

$$d\langle\varphi,\tilde{\zeta}_t^n\rangle = \frac{1}{2}\langle\Delta_n\varphi,\tilde{\zeta}_t^n\rangle dt + \text{mart.}$$

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Central limit theorem

Theorem 2 [Galves, Kipnis, Spohn; Ravishankar '90]

Let the initial density profile ρ_0 be smooth. Then the density fluctuation field

$$\tilde{\zeta}_t^n := \frac{1}{(2n+1)^d} \sum_{x \in \mathbb{T}_n^d} \zeta_t(x)\delta_x$$

converges in $D([0,T], \mathcal{D}')$ to the generalized Ornstein-Uhlenbeck process that solves the linear SPDE

$$d\zeta_t^\infty = \frac{1}{2}\Delta\zeta_t^\infty dt + \nabla \cdot \left(\sqrt{\rho_t^\infty(1-\rho_t^\infty)} dW_t\right)$$

with the centered Gaussian initial condition such that

$$\mathbb{E}\left[\langle \zeta_0, \varphi \rangle^2\right] = \langle \rho_0(1-\rho_0)\varphi, \varphi \rangle$$

Idea of proof

- Consider the semimartingale $\langle \varphi, \tilde{\zeta}_t \rangle$ and its quadratic variation.

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- Convergence of $\langle \text{mart.} \rangle_t$:

$$\eta_t^n \tau_j \eta_t^n = \rho_t^n \tau_j \rho_t^n + \frac{1}{(2n+1)^{d/2}} \left(\rho_t^n \tau_j \zeta_t^n + \zeta_t^n \tau_j \rho_t^n \right) + \frac{1}{(2n+1)^d} \zeta_t^n \tau_j \zeta_t^n,$$

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- Control of $\mathbb{E}\left[\left(\eta_t^n(x) - \rho_t^n(x) \right) \left(\eta_t^n(x+e_j) - \rho_t^n(x+e_j) \right) \right] \to 0$ [Ferrari et al. '91]

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- Use of tightness argument and the Holley-Stroock theory [Holley, Stroock '79]

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$$\tilde{\zeta}_t^n - \zeta_t^\infty = O(?)$$

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$$\sup_{t\in[0,T]} \left|\mathbb{E}f\left(\langle\varphi,\tilde{\zeta}_t^n\rangle\right) - \mathbb{E}f\left(\langle\varphi,\zeta_t^\infty\rangle\right)\right| \to 0$$

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- [Chassagneux, Szpruch, Tse '22]: Weak quantitative propagation of chaos (mean field limit)
- [Kolokoltsov '10] Central limit theorem for the Smoluchovski coagulation model (mean field limit, non-local Smoluchowski's coagulation equation)
- ...

Main tool

Idea of proof: Compare two (time-homogeneous) Markov processes X_t, Y_t taking values in the same state space and $X_0 = Y_0 = x$ using

$$\mathbb{E}F(X_t) - \mathbb{E}F(Y_t) = \int_0^t P_s^X \left(\mathcal{G}^X - \mathcal{G}^Y\right) P_{t-s}^Y F(x) ds,$$
$$= \int_0^t \mathbb{E}\left[\left(\mathcal{G}^X - \mathcal{G}^Y\right) P_{t-s}^Y F(X_s)\right] ds,$$

[see e.g. Ethier, Kurtz '86]

Main result

Theorem 3 [Gess, K. '24]

Let

- the initial density profile $\rho_0 : \mathbb{T}^d \to [0,1]$ be smooth enough,
- η_t^n be SSEP with $\eta_0^n(x) \sim B(\rho_0(x))$ and independent,
- $\rho_t^n = \mathbb{E}\eta_t^n$, $\zeta_t^n = (2n+1)^{d/2}(\eta_t^n - \rho_t^n)$
- ζ_t^∞ solves $d\zeta_t^\infty = \frac{1}{2}\Delta\zeta_t^\infty dt + \nabla \cdot \left(\sqrt{\rho_t^\infty(1-\rho_t^\infty)}dW_t\right)$ with the centered Gaussian initial condition with $\mathbb{E}\left[\langle\zeta_0,\varphi\rangle^2\right] = \langle\rho_0(1-\rho_0)\varphi,\varphi\rangle$

Then for large enough $l \in \mathbb{N}$

$$\sup_{t\in[0,T]} \left|\mathbb{E}f\left(\langle\vec{\varphi},\tilde{\zeta}_t^n\rangle\right) - \mathbb{E}f\left(\langle\vec{\varphi},\zeta_t^\infty\rangle\right)\right| \leq \frac{C}{n^{\frac{d}{2}\wedge 1}} \|f\|_{\mathrm{C}_l^3} \|\vec{\varphi}\|_{\mathrm{C}^l}$$

for all $n \geq 1$, $f \in \mathrm{C}_b^3(\mathbb{R}^m)$ and $\vec{\varphi} \in \left(\mathrm{C}^l(\mathbb{T}^d)\right)^m$.

Main result

Theorem 3 [Gess, K. '24]

Let

- the initial density profile $\rho_0 : \mathbb{T}^d \to [0,1]$ be smooth enough,
- η_t^n be SSEP with $\eta_0^n(x) \sim B(\rho_0(x))$ and independent,
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for all $n \geq 1$, $f \in \mathrm{C}_b^3(\mathbb{R}^m)$ and $\vec{\varphi} \in \left(\mathrm{C}^l(\mathbb{T}^d)\right)^m$.

The rate $\frac{1}{n^{\frac{d}{2}\wedge 1}}$ is optimal: $\frac{1}{n}$ – lattice discretization error, $\frac{1}{n^{\frac{d}{2}}}$ – particle approximation error

Splitting of the problem

Recall

$$\mathbb{E}F(X_t) - \mathbb{E}F(Y_t) = \int_0^t \mathbb{E}\left[\left(\mathcal{G}^X - \mathcal{G}^Y\right) P_{t-s}^Y F(X_s)\right] ds,$$

where $X_0 = Y_0 = x$.

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We consider the Markov processes:

- particle means and fluctuation field: $(\tilde{\rho}_t^n, \tilde{\zeta}_t^n)$
- solution to heat equation and generalized OU process $(\rho_t^\infty, \zeta_t^\infty)$.

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The processes starts from different initial conditions!

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We consider the Markov processes:

- particle means and fluctuation field: $(\tilde{\rho}^n_t, \tilde{\zeta}^n_t)$
- solution to heat equation and generalized OU process $(\rho^\infty_t, \zeta^\infty_t)$.

The processes starts from different initial conditions!

We will compare:

- $(\tilde{\rho}^n_t, \tilde{\zeta}^n_t)$ an $(\rho^{\infty,n}_t, \zeta^{\infty,n}_t)$ **[comparison of dynamics]** where the generalized OU process $(\rho^{\infty,n}_t, \zeta^{\infty,n}_t)$ started from $(\tilde{\rho}^n_0, \tilde{\zeta}^n_0)$;
- $(\rho^{\infty,n}_t, \zeta^{\infty,n}_t)$ and $(\rho^\infty_t, \zeta^\infty_t)$ **[comparison of initial conditions]** (both are defined by the same equation).

Formal comparison of generators

We start from the formal computation for cylindrical functions:

$$F(\tilde{\rho}, \tilde{\zeta}) := f\left(\langle \psi, \tilde{\rho} \rangle, \langle \varphi, \tilde{\zeta} \rangle\right)$$

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Using Taylor's formula, we get:

$$\begin{aligned} \mathcal{G}_n^{FF} F(\tilde{\rho}, \tilde{\zeta}) = &\frac{1}{2} \partial_1 f(\ldots) \langle \Delta_n \varphi, \tilde{\rho} \rangle + \frac{1}{2} \partial_2 f(\ldots) \langle \Delta_n \varphi, \tilde{\zeta} \rangle \\ &+ \frac{1}{4} \partial_2^2 f(\ldots) \left\langle \tau_j \tilde{\eta} + \tilde{\eta} - 2\widetilde{\eta \tau_j \eta}, |\partial_{n,j} \varphi|^2 \right\rangle + O\left(1/n^{\frac{d}{2}+1}\right), \end{aligned}$$

where $\tilde{\zeta} = n^{d/2}(\tilde{\eta} - \tilde{\rho})$.

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where $\tilde{\zeta} = n^{d/2}(\tilde{\eta} - \tilde{\rho})$. Moreover,

$$\begin{aligned}\mathcal{G}^{OU} F(\tilde{\rho}, \tilde{\zeta}) &= \frac{1}{2}\partial_1 f\,(\dots)\,\langle \Delta \varphi, \tilde{\rho} \rangle + \frac{1}{2}\partial_2 f\,(\dots)\,\langle \Delta \varphi, \tilde{\zeta} \rangle \\ &\quad + \frac{1}{4}\partial_2^2 f\,(\dots)\left\langle \tilde{\rho} + \tilde{\rho} - 2\tilde{\rho}^2, |\partial_j \varphi|^2 \right\rangle .\end{aligned}$$

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Using $\eta\tau_j\eta = \rho\tau_j\rho + \frac{1}{(2n+1)^{d/2}}\left(\rho\tau_j\zeta + \zeta\tau_j\rho\right) + \frac{1}{(2n+1)^d}\zeta\tau_j\zeta$, we get

$$\left|(\mathcal{G}_n^{FF} - \mathcal{G}^{OU})F\right| \lesssim \frac{1}{n}\|\psi, \varphi\|_{\mathrm{C}^2}\|f\|_{\mathrm{C}^3} + \frac{1}{n^{d/2}}\left\langle \tilde{\zeta}, \tau_j\rho\,|\partial_{n,j}\varphi|^2\right\rangle + \partial_2^2 f(\langle\ldots\rangle)\left\langle \widetilde{\zeta\tau\zeta}, |\ldots|^2\right\rangle \ldots$$

The difficulties

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- $\sqrt{\rho_t^{\infty,n}(1-\rho_t^{\infty,n})}$ is not well-defined because $\rho_0^{\infty,n} = \tilde{\rho}_0^n$. $\rightsquigarrow \nexists\ \zeta_t^{\infty,n}$

Discrete and continuous Fourier transform

Replace $\tilde{\rho} = \frac{1}{(2n+1)^d} \sum_{x \in \mathbb{T}_n^d} \rho(x)\delta_x$ and $\tilde{\zeta} = \frac{1}{(2n+1)^d} \sum_{x \in \mathbb{T}_n^d} \zeta(x)\delta_x$ by a smooth interpolation.

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- Let $L_2(\mathbb{T}_n^d)$ be the Hilbert space of all functions on $\mathbb{T}_n^d$ with inner product

$$\langle \rho_1, \rho_2 \rangle_n := \frac{1}{(2n+1)^d} \sum_{x \in \mathbb{T}_n^d} \rho_1(x)\rho_2(x)$$

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  – basis vectors on $L_2(\mathbb{T}_n^d)$ and $L_2(\mathbb{T}^d)$, and
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$$L_2(\mathbb{T}_n^d) \ni \rho = \sum_{k \in \mathbb{Z}_n^d} \langle \rho, \varsigma_k \rangle_n \varsigma_k \ \text{ on } \mathbb{T}_n^d, \quad L_2(\mathbb{T}^d) \ni \varphi = \sum_{k \in \mathbb{Z}^d} \langle \varphi, \varsigma_k \rangle \varsigma_k \ \text{ on } \mathbb{T}^d$$

New (smooth) lifting of discrete space

For functions $\rho \in L_2(\mathbb{T}_n^d)$ and $\varphi \in L_2(\mathbb{T}^d)$ define

$$\mathrm{ex}_n\rho := \sum_{k\in\mathbb{Z}_n^d} \langle \rho, \varsigma_k\rangle_n \varsigma_k \quad \text{on } \mathbb{T}^d, \quad \mathrm{pr}_n\varphi := \sum_{k\in\mathbb{Z}_n^d} \langle \varphi, \varsigma_k\rangle \varsigma_k \quad \text{on } \mathbb{T}^d$$

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Basic properties of $\mathrm{ex}_n f$ and $\mathrm{pr}_n g$

- $\mathrm{ex}_n\rho = \rho$ on $\mathbb{T}_n^d$ and $\mathrm{ex}_n\rho \in \mathrm{C}^\infty(\mathbb{T}^d)$
- $\mathrm{pr}_n\varphi$ is well defined on $\mathbb{T}_n^d$ for each $\varphi \in H_J$ for $H_J := \left\{\varphi : \ \|\varphi\|_{H_J}^2 := \sum_{k\in\mathbb{Z}^d}(1+|k|^2)^J\,|\langle\varphi,\varsigma_k\rangle|^2\right\}$, $J \in \mathbb{R}$.
- $\langle \rho_1, \rho_2\rangle_n = \langle \mathrm{ex}_n\rho_1, \mathrm{ex}_n\rho_2\rangle$ and $\langle \rho, \mathrm{pr}_n\varphi\rangle_n = \langle \mathrm{ex}_n\rho, \varphi\rangle$
- $\|\mathrm{pr}_n\varphi - \varphi\|_{H_J} \leq \frac{1}{n^{I-J}}\|\mathrm{pr}_n\varphi - \varphi\|_{H_I}$, $\|\mathrm{ex}_n\varphi - \varphi\|_{H_J} \leq \frac{C}{n}\|\varphi\|_{\mathrm{C}^{J+2+\frac{d}{2}}}, \dots$

$$\langle \varphi, \tilde\rho\rangle = \frac{1}{(2n+1)^d}\sum_{x\in\mathbb{T}_n^d}\varphi(x)\rho(x) = \langle\varphi,\rho\rangle_n$$

$$= \langle \mathrm{pr}_n\varphi, \rho\rangle_n + O(1/n^p) = \langle\varphi, \mathrm{ex}_n\rho\rangle + O(1/n^p)$$

Discrete and continuous calculus

The operators ex_n and pr_n can be nicely combined with $\Delta_n, \partial_{n,j}$ and Δ, ∂. Note that

$$\Delta \varsigma_k = -\lambda_k \varsigma_k \quad \text{and} \quad \Delta_n \varsigma_k = -\lambda_k^n \varsigma,$$

where

$$\lambda_k = 4\pi^2 |k|^2 \quad \text{and} \quad \lambda_k^n = 2(2n+1)^2 \sum_{j=1}^{d} \left[1 - \cos \frac{2\pi k_j}{2n+1}\right] = \lambda_k + O\left(\frac{k_j^4}{n^2}\right)$$

$$\begin{aligned}
\|\mathrm{ex}_n \Delta_n \mathrm{pr}_n g - \mathrm{pr}_n \Delta g\|_{H_J}^2 &= \sum_{k \in \mathbb{Z}^d} (1+|k|^2)^J \left|\langle \mathrm{ex}_n \Delta_n \mathrm{pr}_n g, \varsigma_k\rangle - \langle \mathrm{pr}_n \Delta g, \varsigma_k\rangle\right|^2 \\
&= \sum_{k \in \mathbb{Z}_n^d} (1+|k|^2)^J \left|\langle \Delta_n \mathrm{pr}_n g, \varsigma_k\rangle_n - \langle \Delta g, \varsigma_k\rangle\right|^2 \\
&= \sum_{k \in \mathbb{Z}_n^d} (1+|k|^2)^J \left|\lambda_k^n \langle \mathrm{pr}_n g, \varsigma_k\rangle_n - 4\pi^2 |k|^2 \langle g, \varsigma_k\rangle\right|^2 \\
&= \sum_{k \in \mathbb{Z}_n^d} (1+|k|^2)^J \left|\lambda_k^n - 4\pi^2 |k|^2\right|^2 |\langle g, \varsigma_k\rangle|^2 \le \frac{C}{n^2} \|g\|_{H_{J+2}}^2
\end{aligned}$$

Comparison of generators for smooth interpolation

Now for $F(\mathrm{ex}_n\rho, \mathrm{ex}_n\zeta) := f(\langle\psi, \mathrm{ex}_n\rho\rangle, \langle\varphi, \mathrm{ex}_n\zeta\rangle) = f(\langle \mathrm{pr}_n\psi, \rho\rangle_n, \langle \mathrm{pr}_n\varphi, \zeta\rangle_n)$,

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$$\mathcal{G}_n^{FF} F(\mathrm{ex}_n\rho, \mathrm{ex}_n\zeta) = \frac{1}{2}\partial_1 f(\dots)\langle \Delta_n \mathrm{pr}_n\psi, \rho\rangle_n + \frac{1}{2}\partial_2 f(\dots)\langle \Delta_n \mathrm{pr}_n\varphi, \zeta\rangle_n$$

$$+ \frac{1}{2}\partial_2^2 f(\dots)\sum_{j=1}^{d}\left\langle \left(\partial_{n,j}\mathrm{pr}_n\varphi\right)^2, \left(\tau_j\eta + \eta - 2\eta\tau_j\eta\right)_n \right\rangle + \dots,$$

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$$\left|\mathcal{G}_n^{FF}F - \mathcal{G}_n^{OU}F\right| \lesssim \frac{1}{n^{d/2}}\left\langle \mathrm{ex}_n\zeta, \mathrm{ex}_n\left(\tau_j\rho\left|\partial_{n,j}\varphi\right|^2\right)\right\rangle + \left\langle \mathrm{ex}_n\left(\zeta\tau_j\zeta\right), \partial_2^2 f(\dots)\mathrm{ex}_n\left(\partial_{n,j}\mathrm{pr}_n\varphi\right)^2\right\rangle^2 + \dots$$

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We now need only to control:

- The expectations:

$$\mathbb{E}\left\langle \mathrm{ex}_n\zeta_t^n, \mathrm{ex}_n\left(\tau_j\rho_t^n\left|\partial_{n,j}\varphi\right|^2\right)\right\rangle \le C\|\mathrm{ex}_n\rho_t^n\|_{\mathrm{C}^J}\mathbb{E}\|\mathrm{ex}_n\zeta_t^n\|_{H_{-l}}$$

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$$\left|\mathcal{G}_n^{FF} F - \mathcal{G}_n^{OU} F\right| \lesssim \frac{1}{n^{d/2}}\left\langle \mathrm{ex}_n\zeta, \mathrm{ex}_n\left(\tau_j\rho\left|\partial_{n,j}\varphi\right|^2\right)\right\rangle + \left\langle \mathrm{ex}_n\left(\zeta\tau_j\zeta\right), \partial_2^2 f(\dots)\mathrm{ex}_n\left(\partial_{n,j}\mathrm{pr}_n\varphi\right)^2\right\rangle^2 + ..$$

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- Using the Fourier analysis, the term $\mathbb{E}\left\langle \mathrm{ex}_n\left(\zeta_t^n\tau_j\zeta_t^n\right), \partial_2^2 f(\dots)\left|\dots\right|^2\right\rangle$ can be controlled via

$$\mathbb{E}\prod_{i=1}^{4}\left(\eta_t^n(x_i) - \rho_t^n(x_i)\right) \lesssim \frac{1}{n}$$

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$$\mathcal{G}_n^{FF} F(\mathrm{ex}_n\rho, \mathrm{ex}_n\zeta) = \frac{1}{2}\partial_1 f(\ldots)\langle \mathrm{ex}_n\Delta_n \mathrm{pr}_n\psi, \mathrm{ex}_n\rho\rangle + \frac{1}{2}\partial_2 f(\ldots)\langle \mathrm{ex}_n\Delta_n \mathrm{pr}_n\varphi, \mathrm{ex}_n\zeta\rangle + \frac{1}{2}\partial_2^2 f(\ldots)\sum_{j=1}^{d}\left\langle \mathrm{ex}_n\left(\partial_{n,j}\mathrm{pr}_n\varphi\right)^2, \mathrm{ex}_n\left(\tau_j\eta + \eta - 2\eta\tau_j\eta\right)\right\rangle + \ldots,$$

$$\left|\mathcal{G}_n^{FF}F - \mathcal{G}_n^{OU}F\right| \lesssim \frac{1}{n^{d/2}}\left\langle \mathrm{ex}_n\zeta, \mathrm{ex}_n\left(\tau_j\rho\left|\partial_{n,j}\varphi\right|^2\right)\right\rangle + \left\langle \mathrm{ex}_n\left(\zeta\tau_j\zeta\right), \partial_2^2 f(\ldots)\mathrm{ex}_n\left(\partial_{n,j}\mathrm{pr}_n\varphi\right)^2\right\rangle^2 + \ldots$$

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$$\mathbb{E}\prod_{i=1}^{4}\left(\eta_t^n(x_i) - \rho_t^n(x_i)\right) \lesssim \frac{1}{n}$$
- We can compare generators on smooth enough functions on Sobolev spaces

Frechet differentiable function on H_J

Let $D^m F(g)$ denoted the m-Frechet derivative of F that is a continuous multilinear operator on $(H_J)^m$.

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Example: For $F(g) = f(\langle \varphi, g \rangle)$, we get for $h, h_1, h_2 \in H_J$

$$DF(g)[h] = \left\langle f'(\langle \varphi, g \rangle)\, \varphi, h \right\rangle \quad \text{and} \quad D^2 F(g)[h_1, h_2] = \left\langle f''(\langle \varphi, g \rangle)\, \varphi \otimes \varphi, h_1 \otimes h_2 \right\rangle.$$

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In particular, for $F(\mathrm{ex}_n \rho, \mathrm{ex}_n \zeta) := f(\langle \psi, \mathrm{ex}_n \rho \rangle, \langle \varphi, \mathrm{ex}_n \zeta \rangle)$

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$$+ \sum_{j=1}^{d}\left\langle \mathrm{ex}_n \mathrm{Tr}\left(\partial_{n,j}^{\otimes 2}\mathrm{pr}_n^{\otimes 2} D_2^2 F(\hat{\eta})\right), \mathrm{ex}_n\left(\tau_j\eta + \eta - 2\eta\tau_j\eta\right)\right\rangle + \ldots,$$

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$\rightsquigarrow$ **The expansion holds for each function** $F \in \mathrm{C}^{1,3}(H_J \times H_{-I})$**!**

Differentiability of $P_t^{OU} F(\mathrm{ex}_n \rho, \mathrm{ex}_n \zeta)$

A solution to

$$d\rho_t^\infty = \frac{1}{2}\Delta\rho_t^\infty dt$$
$$d\zeta_t^\infty = \frac{1}{2}\Delta\zeta_t^\infty dt + \nabla \cdot \left(\sqrt{\rho_t^\infty(1-\rho_t^\infty)}dW_t\right)$$

exists for all $\rho_0^\infty \in L_2(\mathbb{T}^d; [0,1])$ and $\zeta_0^\infty \in H_{-l}$ for $l > \frac{d}{2} + 1$.

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For $F \in \mathrm{C}(H_{-l})$ (e.g. $F(g) = f(\langle \varphi, g \rangle)$, $\varphi \in \mathrm{C}^{\lceil l \rceil}$) define $U_t(\rho_0^\infty, \zeta_0^\infty) := \mathbb{E} F(\zeta_t^\infty)$

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Proposition 1 [Gess, K. '24]

Let $l > \frac{d}{2}+1$ and $F \in \mathrm{C}_b^{2,4}(H_{-l})$. Then $U_t \in \mathrm{C}_b^{1,3}(H_J \times H_{-l})$ for $J > \frac{d}{2}$. Moreover,

$$D_1 U_t(\rho_0^\infty, \zeta_0^\infty)[h] = \frac{1}{2}\mathbb{E}\left[D^2F(\zeta_t^\infty) : DV_t(\rho_0^\infty)[h]\right]$$

with

$$V_t(\rho_0^\infty)(\varphi,\psi) = \mathrm{Cov}\left(\langle\varphi,\zeta_t^\infty\rangle, \langle\psi,\zeta_t^\infty\rangle\right)$$
$$= \frac{1}{2}\int_0^t \left\langle \nabla P_{t-s}^{HE}\varphi\cdot\nabla P_{t-s}^{HE}\psi, \rho_s^\infty(1-\rho_s^\infty)\right\rangle ds$$

Idea of proof

We show the (uniform in n) differentiability

$$U^n(\rho_0^\infty) := \mathbb{E} F\left(\mathrm{pr}_n \zeta_t^\infty\right)$$

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Setting $\xi^n := \left(\langle \zeta_t^\infty, \varsigma_k\rangle - m_k\right)_{k\in\mathbb{Z}_n^d}$ for $m_k^n = \mathbb{E}\langle \zeta_t^\infty, \varsigma_k\rangle$, we can represent U^n as follows

$$U^n(\rho_0^\infty) = \mathbb{E}f_n\left(m^n + \xi^n\right) = \mathbb{E}f_n\Big(m^n + \sqrt{V^n(\rho_0^\infty)}\tilde{\xi}^n\Big),$$

where $V^n(\rho_0^\infty) := \left(V_t(\rho_0^\infty)[\varsigma_k, \varsigma_l]\right)_{k,l\in\mathbb{Z}_n^d}$.

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Reason for differentiability: $F(z) = f(m + \sqrt{z}\tilde{\xi})$:

$$\begin{aligned}
\frac{\partial}{\partial z}F(x,z) &= \frac{\partial}{\partial z}\int_{\mathbb{R}} f\left(m+\sqrt{z}y\right)\frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}dy \\
&= \int_{\mathbb{R}} f'\left(m+\sqrt{z}y\right)\frac{y}{2\sqrt{2\pi z}}e^{-\frac{y^2}{2}}dy \\
&= -\frac{1}{2}\int_{\mathbb{R}} f'\left(m+\sqrt{z}y\right)\frac{1}{\sqrt{2\pi z}}de^{-\frac{y^2}{2}} \\
&= \frac{1}{2}\int_{\mathbb{R}} f''\left(m+\sqrt{z}y\right)\frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}dy.
\end{aligned}$$

Comparison of dynamics

Recall that (ρ_t^n, ζ_t^n) is the mean process together with the fluctuation field of SSEP.

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started from $(\mathrm{ex}_n\rho_0^n, \mathrm{ex}_n\zeta_0^n)$.

Then for each I, J large enough and $F \in \mathrm{C}_b(H_J \times H_{-I})$

$$|\mathbb{E}F(\mathrm{ex}_n\rho_t^n, \mathrm{ex}_n\zeta_t^n) - \mathbb{E}F(\rho_t^{\infty,n}, \zeta_t^{\infty,n})| \leq \int_0^t \left|\mathbb{E}\left[\left(\mathcal{G}^{FF} - \mathcal{G}^{OU}\right) P_{t-s}^{OU} F(\mathrm{ex}_n\rho_t^n, \mathrm{ex}_n\zeta_t^n)\right]\right| ds$$

$$\leq C\int_0^t \left(\frac{1}{n}\|F\|_{\mathrm{C}^{1,3}} + \frac{1}{n^{d/2}}\|\mathrm{ex}_n\rho_s^n\|_{\mathrm{C}^J}\mathbb{E}\|\mathrm{ex}_n\zeta_s^n\|_{H_{-I}} + \mathbb{E}\left\langle \mathrm{ex}_n\left(\zeta_s^n\tau_j\zeta_s^n\right), \ldots\right\rangle^2 + \ldots\right) ds$$

Berry-Esseen bound for the initial fluctuations

- It remains only to compare

$$\mathbb{E}F(\rho_t^{\infty,n},\zeta_t^{\infty,n})-\mathbb{E}F(\rho_t^{\infty},\zeta_t^{\infty})=P_t^{OU}F(\mathrm{ex}_n\rho_0^n,\mathrm{ex}_n\zeta_0^n)-P_t^{OU}F(\rho_0,\zeta_0)$$

  where ρ_t^{∞}started from the initial profile ρ_0 and ζ_t started from the centered Gaussian distribution with

$$\mathbb{E}\langle\zeta_0,\varphi\rangle^2=\langle\rho_0(1-\rho_0)\varphi,\varphi\rangle.$$

- It is enough to compare only

$$\mathbb{E}F(\mathrm{ex}_n\zeta_0^n)-\mathbb{E}F(\mathrm{pr}_n\zeta_0),$$

  for $F\in\mathrm{C}^3(H_{-l})$, where

$$\mathrm{ex}_n\zeta_0^n:=\sum_{k\in\mathbb{Z}_n^d}\langle\zeta_0^n,\varsigma_k\rangle_n\varsigma_k,\quad \mathrm{pr}_n\zeta_0:=\sum_{k\in\mathbb{Z}_n^d}\langle\zeta_0,\varsigma_k\rangle\varsigma_k$$

- Is enough to compare for $f\in\mathrm{C}^3\left(\mathbb{R}^{\mathbb{Z}_n^d}\right)$

$$\mathbb{E}f\left(\left((1+|k|^2)^{-l/2}\langle\zeta_0^n,\varsigma_k\rangle_n\right)_{k\in\mathbb{Z}_n^d}\right)-\mathbb{E}f\left(\left((1+|k|^2)^{-l/2}\langle\zeta_0,\varsigma_k\rangle\right)_{k\in\mathbb{Z}_n^d}\right).$$

- Apply multidimensional Berry-Essen theorem [Meckes '09]

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Thank you!