A Quantitative Central Limit Theorem for Simple Symmetric Exclusion Process

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joint work with Benjamin Gess

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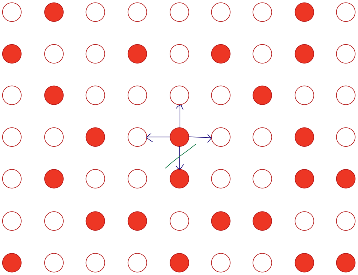
Simple symmetric exclusion process

On the *d*-dim discrete torus

$$\mathbb{T}_n^d := \left\{ \frac{k}{2n+1} : \ k \in \mathbb{Z}_n^d := \{-n, \dots, n\}^d \right\} \subset \mathbb{T}^d = (\mathbb{R}/\{\mathbb{Z} - 1/2\})^d$$

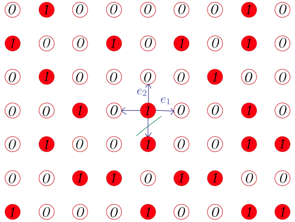
we consider a **Simple Symmetric Exclusion Process** (SSEP)



State space and generator

Particle configuration $\eta \in \{0,1\}^{\mathbb{T}_n^d}$:

$$\eta(x) = 0 \Leftrightarrow \text{side } x \text{ is empty}$$
$$\eta(x) = 1 \Leftrightarrow \text{side } x \text{ is occupied}$$



$$\eta^{x\leftrightarrow y}(z) = \begin{cases} \eta(z), & z \neq x, y, \\ \eta(y), & z = x, \\ \eta(x), & z = y, \end{cases}$$

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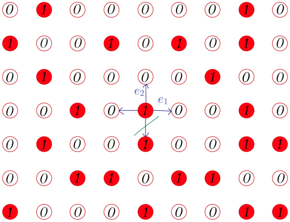

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$$\mathcal{G}_n^{EP} F(\eta) := \frac{(2n+1)^2}{2} \sum_{j=1}^{d} \sum_{x \in \mathbb{T}_n} \left[F(\eta^{x\leftrightarrow x+e_j}) - F(\eta) \right]$$

SSEP is already parabolically rescaled: space $\sim \frac{1}{2n+1}$ time $\sim (n+1)^2$!

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Equilibrium SSEP:

For fixed $\rho_0 \in [0,1]$, the measure $B(\rho_0)^{\otimes \mathbb{T}_n^d}$ is invariant for SSEP η_t^n:

If $\eta_0^n(x) \sim B(\rho_0)$, $x \in \mathbb{T}_n^d$, are independent then $\eta_t^n(x) \sim B(\rho_0)$, $x \in \mathbb{T}_n^d$, are independent.

$\rightsquigarrow \mathbb{E}\eta_t^n(x) = \rho_0, \ \mathrm{Var}\, \eta_t^n = \rho_0(1-\rho_0)$.

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What is $\mathbb{E}\eta_t^n(x)$?

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Set $\rho_t^n(x) := \mathbb{E}\eta_t^n(x)$.

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Then

$$d\rho_t^n(x) = \mathbb{E}\mathcal{G}_n^{EP}\eta_t^n(x)dt = \mathbb{E}\frac{(2n+1)^2}{2}\sum_{j=1}^{d}\sum_{y\in\mathbb{T}_n}\left[\eta^{y\leftrightarrow y+e_j}(x) - \eta(x)\right]dt$$

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In particular, the empirical distribution

$$\tilde{\rho}_t^n := \frac{1}{(2n+1)^d}\sum_{x\in\mathbb{T}_n^d}\rho_t(x)\delta_x,$$

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Law of large numbers

Theorem [see e.g. in Kipnis, Landim '99]

Let $\rho_0 : \mathbb{T}^d \to [0,1]$ be an initial density profile and $\eta_0^n(x) \sim B(\rho_0(x))$ be independent. Then

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$$= \frac{1}{2} f'(\langle \varphi, \tilde{\eta} \rangle) \langle \Delta_n \varphi, \tilde{\eta} \rangle + \frac{1}{4(2n+1)^d} f''(\langle \varphi, \tilde{\eta} \rangle) \left\langle \tau_j \tilde{\eta} + \tilde{\eta} - 2\widetilde{\eta \tau_j \eta}, \left| \partial_{n,j} \varphi \right|^2 \right\rangle + \dots,$$

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where $\tau_j\eta(x) := \eta(x+e_j)$. $\rightsquigarrow$ $d\langle\varphi,\tilde{\eta}_t^n\rangle = \langle\Delta_n\varphi,\tilde{\eta}_t^n\rangle dt + \text{mart.}$

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Density fluctuation field and CLT

We now consider the fluctuations of the SSEP around its mean:

$$\zeta_t^n(x) := (2n+1)^{d/2} \left(\eta_t^n(x) - \rho_t^n(x)\right).$$

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The generator of

$$\tilde{\zeta}_t^n := \frac{1}{(2n+1)^d} \sum_{x \in \mathbb{T}_n^d} \zeta_t(x) \delta_x$$

can be expanded as follows

$$\begin{aligned} \mathcal{G}_n^{FF} f \left(\langle \varphi, \tilde{\zeta} \rangle\right) &= \mathcal{G}_n^{EP} f \left((2n+1)^{d/2} \langle \varphi, \tilde{\eta} - \tilde{\rho} \rangle\right) \\ &= \frac{1}{2} f' \left(\langle \varphi, \tilde{\zeta} \rangle\right) \langle \Delta_n \varphi, \tilde{\zeta} \rangle + \frac{(2n+1)^d}{4(2n+1)^d} f'' \left(\langle \varphi, \tilde{\zeta} \rangle\right) \left\langle \tau_j \tilde{\eta} + \tilde{\eta} - 2\widetilde{\eta \tau_j \eta}, |\partial_{n,j} \varphi|^2 \right\rangle \\ &+ O\left(1/n^{\frac{d}{2}+1}\right) \end{aligned}$$

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Again

$$d\langle \varphi, \tilde{\zeta}_t^n \rangle = \frac{1}{2} \langle \Delta_n \varphi, \tilde{\zeta}_t^n \rangle dt + \text{mart.}$$

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Central limit theorem

Theorem 2 [Galves, Kipnis, Spohn; Ravishankar '90]

Let the initial density profile ρ_0 be smooth. Then the density fluctuation field

$$\tilde{\zeta}_t^n := \frac{1}{(2n+1)^d} \sum_{x \in \mathbb{T}_n^d} \zeta_t(x) \delta_x$$

converges in $D([0,T], \mathcal{D}')$ to the generalized Ornstein-Uhlenbeck process that solves the linear SPDE

$$d\zeta_t^\infty = \frac{1}{2} \Delta \zeta_t^\infty dt + \nabla \cdot \left(\sqrt{\rho_t^\infty (1 - \rho_t^\infty)} dW_t \right)$$

with the centered Gaussian initial condition such that

$$\mathbb{E}\left[\langle \zeta_0, \varphi \rangle^2 \right] = \langle \rho_0 (1 - \rho_0) \varphi, \varphi \rangle$$

Idea of proof

- Consider the semimartingale $\langle \varphi, \tilde{\zeta}_t \rangle$ and its quadratic variation.

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- Use of tightness argument and the Holley-Stroock theory [Holley, Stroock '79]

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$$\tilde{\zeta}_t^n - \zeta_t^\infty = O(?)$$

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- [Kolokoltsov '10] Central limit theorem for the Smoluchovski coagulation model (mean field limit, non-local Smoluchowski's coagulation equation)
- ...

Main tool

Idea of proof: Compare two (time-homogeneous) Markov processes X_t, Y_t taking values in the same state space and $X_0 = Y_0 = x$ using

$$\mathbb{E}F(X_t) - \mathbb{E}F(Y_t) = \int_0^t P_s^X \left(\mathcal{G}^X - \mathcal{G}^Y\right) P_{t-s}^Y F(x) ds,$$
$$= \int_0^t \mathbb{E}\left[\left(\mathcal{G}^X - \mathcal{G}^Y\right) P_{t-s}^Y F(X_s)\right] ds,$$

[see e.g. Ethier, Kurtz '86]

Main result

Theorem 3 [Gess, K. '24]

Let

- the initial density profile $\rho_0 : \mathbb{T}^d \to [0,1]$ be smooth enough,
- η_t^n be SSEP with $\eta_0^n(x) \sim B(\rho_0(x))$ and independent,
- $\rho_t^n = \mathbb{E}\eta_t^n$, $\zeta_t^n = (2n+1)^{d/2}(\eta_t^n - \rho_t^n)$
- ζ_t^∞ solves $d\zeta_t^\infty = \frac{1}{2}\Delta\zeta_t^\infty dt + \nabla \cdot \left(\sqrt{\rho_t^\infty(1-\rho_t^\infty)}dW_t\right)$ with the centered Gaussian initial condition with $\mathbb{E}\left[\langle\zeta_0, \varphi\rangle^2\right] = \langle\rho_0(1-\rho_0)\varphi, \varphi\rangle$

Then for large enough $l \in \mathbb{N}$

$$\sup_{t\in[0,T]} \left|\mathbb{E}f\left(\langle\varphi, \tilde{\zeta}_t^n\rangle\right) - \mathbb{E}f\left(\langle\varphi, \zeta_t^\infty\rangle\right)\right| \leq \frac{C}{n^{\frac{d}{2}\wedge 1}} \|f\|_{\mathrm{C}_l^3} \|\vec{\varphi}\|_{\mathrm{C}^l}$$

for all $n \geq 1$, $f \in \mathrm{C}_b^3(\mathbb{R})$ and $\varphi \in \mathrm{C}^l(\mathbb{T}^d)$.

Main result

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for all $n \geq 1$, $f \in \mathrm{C}_b^3(\mathbb{R})$ and $\varphi \in \mathrm{C}^l(\mathbb{T}^d)$.

The rate $\frac{1}{n^{\frac{d}{2}\wedge 1}}$ is optimal: $\frac{1}{n}$ – lattice discretization error, $\frac{1}{n^{\frac{d}{2}}}$ – particle approximation error

Splitting of the problem

Recall

$$\mathbb{E}F(X_t) - \mathbb{E}F(Y_t) = \int_0^t \mathbb{E}\left[\left(\mathcal{G}^X - \mathcal{G}^Y\right) P_{t-s}^Y F(X_s)\right] ds,$$

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We consider the Markov processes:

- particle means and fluctuation field: $(\tilde{\rho}_t^n, \tilde{\zeta}_t^n)$
- solution to heat equation and generalized OU process $(\rho_t^\infty, \zeta_t^\infty)$.

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The processes starts from different initial conditions!

We will compare:

- $(\tilde{\rho}^n_t, \tilde{\zeta}^n_t)$ an $(\rho^{\infty,n}_t, \zeta^{\infty,n}_t)$ **[comparison of dynamics]**
  where the generalized OU process $(\rho^{\infty,n}_t, \zeta^{\infty,n}_t)$ started from $(\tilde{\rho}^n_0, \tilde{\zeta}^n_0)$;
- $(\rho^{\infty,n}_t, \zeta^{\infty,n}_t)$ and $(\rho^\infty_t, \zeta^\infty_t)$ **[comparison of initial conditions]**
  (both are defined by the same equation).

Formal comparison of generators

We start from the formal computation for cylindrical functions:

$$F(\tilde{\rho}, \tilde{\zeta}) := f\left(\langle \psi, \tilde{\rho} \rangle, \langle \varphi, \tilde{\zeta} \rangle\right)$$

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Using Taylor's formula, we get:

$$\begin{aligned}\mathcal{G}_n^{FF} F(\tilde{\rho}, \tilde{\zeta}) &= \frac{1}{2}\partial_1 f\,(\dots)\,\langle \Delta_n \varphi, \tilde{\rho} \rangle + \frac{1}{2}\partial_2 f\,(\dots)\,\langle \Delta_n \varphi, \tilde{\zeta} \rangle \\ &\quad + \frac{1}{4}\partial_2^2 f\,(\dots)\left\langle \tau_j \tilde{\eta} + \tilde{\eta} - 2\widetilde{\eta \tau_j \eta}, |\partial_{n,j}\varphi|^2 \right\rangle + O\left(1/n^{\frac{d}{2}+1}\right),\end{aligned}$$

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Using $\eta \tau_j \eta = \rho \tau_j \rho + \frac{1}{(2n+1)^{d/2}}(\rho \tau_j \zeta + \zeta \tau_j \rho) + \frac{1}{(2n+1)^d}\zeta \tau_j \zeta$, we get

$$\left|(\mathcal{G}_n^{FF} - \mathcal{G}^{OU})F\right| \lesssim \frac{1}{n}\|\psi, \varphi\|_{C^2}\|f\|_{C^3} + \frac{1}{n^{d/2}}\left\langle \tilde{\zeta}, \tau_j \rho \, |\partial_{n,j}\varphi|^2 \right\rangle + \partial_2^2 f(\langle \ldots \rangle)\left\langle \widetilde{\zeta \tau \zeta}, |\ldots|^2 \right\rangle \ldots$$

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- $\sqrt{\rho_t^{\infty,n}(1-\rho_t^{\infty,n})}$ is not well-defined because $\rho_0^{\infty,n} = \tilde{\rho}_0^n$. $\rightsquigarrow \nexists\ \zeta_t^{\infty,n}$

Discrete and continuous Fourier transform

Replace $\tilde{\rho} = \frac{1}{(2n+1)^d} \sum_{x \in \mathbb{T}_n^d} \rho(x)\delta_x$ and $\tilde{\zeta} = \frac{1}{(2n+1)^d} \sum_{x \in \mathbb{T}_n^d} \zeta(x)\delta_x$ by a smooth interpolation.

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- Let $L_2(\mathbb{T}_n^d)$ be the Hilbert space of all functions on $\mathbb{T}_n^d$ with inner product

$$\langle \rho_1, \rho_2 \rangle_n := \frac{1}{(2n+1)^d} \sum_{x \in \mathbb{T}_n^d} \rho_1(x)\rho_2(x)$$

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- $\varsigma_k(x) = e^{2\pi i k \cdot x}$, $k \in \mathbb{Z}^d$, $x \in \mathbb{T}^d \supset \mathbb{T}_n^d$
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$$L_2(\mathbb{T}_n^d) \ni \rho = \sum_{k \in \mathbb{Z}_n^d} \langle \rho, \varsigma_k \rangle_n \varsigma_k \quad \text{on } \mathbb{T}_n^d, \qquad L_2(\mathbb{T}^d) \ni \varphi = \sum_{k \in \mathbb{Z}^d} \langle \varphi, \varsigma_k \rangle \varsigma_k \quad \text{on } \mathbb{T}^d$$

New (smooth) lifting of discrete space

For functions $\rho \in L_2(\mathbb{T}_n^d)$ and $\varphi \in L_2(\mathbb{T}^d)$ define

$$\mathrm{ex}_n \rho := \sum_{k \in \mathbb{Z}_n^d} \langle \rho, \varsigma_k \rangle_n \varsigma_k \quad \text{on } \mathbb{T}^d, \quad \mathrm{pr}_n \varphi := \sum_{k \in \mathbb{Z}_n^d} \langle \varphi, \varsigma_k \rangle \varsigma_k \quad \text{on } \mathbb{T}^d$$

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Basic properties of $\mathrm{ex}_n f$ and $\mathrm{pr}_n g$

- $\mathrm{ex}_n\rho = \rho$ on $\mathbb{T}_n^d$ and $\mathrm{ex}_n\rho \in \mathrm{C}^\infty(\mathbb{T}^d)$
- $\mathrm{pr}_n\varphi$ is well defined on $\mathbb{T}_n^d$ for each $\varphi \in H_J$ for $H_J := \left\{\varphi : \ \|\varphi\|_{H_J}^2 := \sum_{k\in\mathbb{Z}^d} (1+|k|^2)^J \left|\langle \varphi, \varsigma_k\rangle\right|^2\right\}$, $J \in \mathbb{R}$.
- $\langle \rho_1, \rho_2\rangle_n = \langle \mathrm{ex}_n\rho_1, \mathrm{ex}_n\rho_2\rangle$ and $\langle \rho, \mathrm{pr}_n\varphi\rangle_n = \langle \mathrm{ex}_n\rho, \varphi\rangle$
- $\|\mathrm{pr}_n\varphi - \varphi\|_{H_J} \le \frac{1}{n^{I-J}} \|\mathrm{pr}_n\varphi - \varphi\|_{H_I}$, $\|\mathrm{ex}_n\varphi - \varphi\|_{H_J} \le \frac{C}{n} \|\varphi\|_{\mathrm{C}^{J+2+\frac{d}{2}}}, \dots$
- $\|\mathrm{ex}_n\Delta_n\mathrm{pr}_n\varphi - \Delta\varphi\|_{H_J} \le \frac{C}{n}\|\varphi\|_{H_{J+2}}$, ...

$$\langle \varphi, \tilde{\rho}\rangle = \frac{1}{(2n+1)^d} \sum_{x\in\mathbb{T}_n^d} \varphi(x)\rho(x) = \langle \varphi, \rho\rangle_n$$

$$= \langle \mathrm{pr}_n\varphi, \rho\rangle_n + O(1/n^p) = \langle \varphi, \mathrm{ex}_n\rho\rangle + O(1/n^p)$$

Comparison of generators for smooth interpolation

Now for $F(\mathrm{ex}_n\rho, \mathrm{ex}_n\zeta) := f\left(\langle\psi, \mathrm{ex}_n\rho\rangle, \langle\varphi, \mathrm{ex}_n\zeta\rangle\right) = f\left(\langle \mathrm{pr}_n\psi, \rho\rangle_n, \langle \mathrm{pr}_n\varphi, \zeta\rangle_n\right)$,

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$$+ \frac{1}{2}\partial_2^2 f(\ldots)\sum_{j=1}^{d}\left\langle\left(\partial_{n,j}\mathrm{pr}_n\varphi\right)^2, \left(\tau_j\eta + \eta - 2\eta\tau_j\eta\right)\right\rangle_n + \ldots,$$

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$$\mathbb{E}\left\langle \mathrm{ex}_n\zeta_t^n, \mathrm{ex}_n\left(\tau_j\rho_t^n\left|\partial_{n,j}\varphi\right|^2\right)\right\rangle \leq C\|\mathrm{ex}_n\rho_t^n\|_{\mathrm{C}^J}\mathbb{E}\|\mathrm{ex}_n\zeta_t^n\|_{H_{-l}}$$

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- Using the Fourier analysis, the term $\mathbb{E}\left\langle \mathrm{ex}_n\left(\zeta_t^n\tau_j\zeta_t^n\right), \partial_2^2 f(\ldots)|\ldots|^2\right\rangle$ can be controlled via

$$\mathbb{E}\prod_{i=1}^{4}\left(\eta_t^n(x_i) - \rho_t^n(x_i)\right) \lesssim \frac{1}{n}$$

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$$\mathcal{G}_n^{FF} F(\mathrm{ex}_n\rho, \mathrm{ex}_n\zeta) = \frac{1}{2}\partial_1 f(\dots)\langle \mathrm{ex}_n\Delta_n \mathrm{pr}_n\psi, \mathrm{ex}_n\rho\rangle + \frac{1}{2}\partial_2 f(\dots)\langle \mathrm{ex}_n\Delta_n \mathrm{pr}_n\varphi, \mathrm{ex}_n\zeta\rangle$$
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- Using the Fourier analysis, the term $\mathbb{E}\left\langle \mathrm{ex}_n\left(\zeta_t^n\tau_j\zeta_t^n\right), \partial_2^2 f(\dots)\left|\dots\right|^2\right\rangle$ can be controlled via
$$\mathbb{E}\prod_{i=1}^{4}\left(\eta_t^n(x_i) - \rho_t^n(x_i)\right) \lesssim \frac{1}{n}$$
- We can compare generators on smooth enough functions on Sobolev spaces

Frechet differentiable function on H_J

Let $D^m F(g)$ denoted the m-Frechet derivative of F that is a continuous multilinear operator on $(H_J)^m$.

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Example: For $F(g) = f(\langle \varphi, g \rangle)$, we get for $h, h_1, h_2 \in H_J$

$$DF(g)[h] = \left\langle f'(\langle \varphi, g \rangle)\, \varphi, h \right\rangle \quad \text{and} \quad D^2 F(g)[h_1, h_2] = \left\langle f''(\langle \varphi, g \rangle)\, \varphi \otimes \varphi, h_1 \otimes h_2 \right\rangle.$$

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$$+ \sum_{j=1}^{d} \left\langle \mathrm{ex}_n \mathrm{Tr}\left(\partial_{n,j}^{\otimes 2} \mathrm{pr}_n^{\otimes 2} D_2^2 F(\hat{\eta})\right), \mathrm{ex}_n\left(\tau_j\eta + \eta - 2\eta\tau_j\eta\right)\right\rangle + \ldots,$$

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$\rightsquigarrow$ The expansion holds for each function $F \in \mathrm{C}^{1,3}(H_J \times H_{-l})$!

Differentiability of $P_t^{OU}F(\mathrm{ex}_n\rho, \mathrm{ex}_n\zeta)$

A solution to

$$d\rho_t^\infty = \frac{1}{2}\Delta\rho_t^\infty dt$$
$$d\zeta_t^\infty = \frac{1}{2}\Delta\zeta_t^\infty dt + \nabla\cdot\left(\sqrt{\rho_t^\infty(1-\rho_t^\infty)}dW_t\right)$$

exists for all $\rho_0^\infty \in L_2(\mathbb{T}^d;[0,1])$ and $\zeta_0^\infty \in H_{-l}$ for $l > \frac{d}{2}+1$.

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For $F \in \mathrm{C}(H_{-l})$ (e.g. $F(g) = f(\langle \varphi, g\rangle)$, $\varphi \in \mathrm{C}^{\lceil l \rceil}$) define $U_t(\rho_0^\infty, \zeta_0^\infty) := \mathbb{E}F\left(\zeta_t^\infty\right)$

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For $F \in \mathrm{C}(H_{-I})$ (e.g. $F(g) = f(\langle \varphi, g \rangle)$, $\varphi \in \mathrm{C}^{\lceil I \rceil}$) define $U_t(\rho_0^\infty, \zeta_0^\infty) := \mathbb{E} F\left(\zeta_t^\infty\right)$

Proposition 1 [Gess, K. '24]

Let $I > \frac{d}{2} + 1$ and $F \in \mathrm{C}_b^{2,4}(H_{-I})$. Then $U_t \in \mathrm{C}_b^{1,3}(H_J \times H_{-I})$ for $J > \frac{d}{2}$. Moreover,

$$D_1 U_t(\rho_0^\infty, \zeta_0^\infty)[h] = \frac{1}{2}\mathbb{E}\left[D^2 F(\zeta_t^\infty) : DV_t(\rho_0^\infty)[h] \right]$$

with

$$V_t(\rho_0^\infty)(\varphi, \psi) = \mathrm{Cov}\left(\langle \varphi, \zeta_t^\infty \rangle, \langle \psi, \zeta_t^\infty \rangle \right)$$
$$= \frac{1}{2}\int_0^t \left\langle \nabla P_{t-s}^{HE}\varphi \cdot \nabla P_{t-s}^{HE}\psi, \rho_s^\infty \left(1 - \rho_s^\infty\right) \right\rangle ds$$

Comparison of dynamics

Recall that (ρ_t^n, ζ_t^n) is the mean process together with the fluctuation field of SSEP.

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$(\rho_t^{\infty,n}, \zeta_t^{\infty,n})$ is a solution to

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$(\rho^{\infty,n}_t, \zeta^{\infty,n}_t)$ is a solution to

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started from $(\mathrm{ex}_n\rho^n_0, \mathrm{ex}_n\zeta^n_0)$.

Then for each I, J large enough and $F \in \mathrm{C}_b(H_J \times H_{-I})$

$$|\mathbb{E}F(\mathrm{ex}_n\rho^n_t, \mathrm{ex}_n\zeta^n_t) - \mathbb{E}F(\rho^{\infty,n}_t, \zeta^{\infty,n}_t)| \leq \int_0^t \left|\mathbb{E}\left[\left(\mathcal{G}^{FF} - \mathcal{G}^{OU}\right) P^{OU}_{t-s}F(\mathrm{ex}_n\rho^n_t, \mathrm{ex}_n\zeta^n_t)\right]\right| ds$$

$$\leq C\int_0^t \left(\frac{1}{n}\|F\|_{\mathrm{C}^{1,3}} + \frac{1}{n^{d/2}}\|\mathrm{ex}_n\rho^n_s\|_{\mathrm{C}^J}\mathbb{E}\|\mathrm{ex}_n\zeta^n_s\|_{H_{-I}} + \mathbb{E}\left\langle \mathrm{ex}_n\left(\zeta^n_s\tau_j\zeta^n_s\right), \dots\right\rangle^2 + \dots\right) ds$$

Berry-Esseen bound for the initial fluctuations

- It remains only to compare

$$\mathbb{E}F(\rho_t^{\infty,n},\zeta_t^{\infty,n}) - \mathbb{E}F(\rho_t^{\infty},\zeta_t^{\infty}) = P_t^{OU}F(\mathrm{ex}_n\rho_0^n,\mathrm{ex}_n\zeta_0^n) - P_t^{OU}F(\rho_0,\zeta_0)$$

  where ρ_t^{∞}started from the initial profile ρ_0 and ζ_t started from the centered Gaussian distribution with

$$\mathbb{E}\langle\zeta_0,\varphi\rangle^2 = \langle\rho_0(1-\rho_0)\varphi,\varphi\rangle.$$

- It is enough to compare only

$$\mathbb{E}F(\mathrm{ex}_n\zeta_0^n) - \mathbb{E}F(\mathrm{pr}_n\zeta_0),$$

  for $F \in \mathrm{C}^3(H_{-l})$, where

$$\mathrm{ex}_n\zeta_0^n := \sum_{k\in\mathbb{Z}_n^d}\langle\zeta_0^n,\varsigma_k\rangle_n\varsigma_k, \quad \mathrm{pr}_n\zeta_0 := \sum_{k\in\mathbb{Z}_n^d}\langle\zeta_0,\varsigma_k\rangle\varsigma_k$$

- Is enough to compare for $f \in \mathrm{C}^3\left(\mathbb{R}^{\mathbb{Z}_n^d}\right)$

$$\mathbb{E}f\left(\left((1+|k|^2)^{-l/2}\langle\zeta_0^n,\varsigma_k\rangle_n\right)_{k\in\mathbb{Z}_n^d}\right) - \mathbb{E}f\left(\left((1+|k|^2)^{-l/2}\langle\zeta_0,\varsigma_k\rangle\right)_{k\in\mathbb{Z}_n^d}\right).$$

- Apply multidimensional Berry-Essen theorem [Meckes '09]

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Thank you!