Systems of massive diffusion particles with singular interaction

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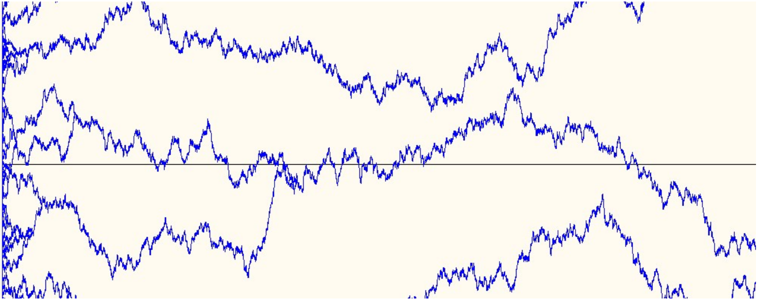
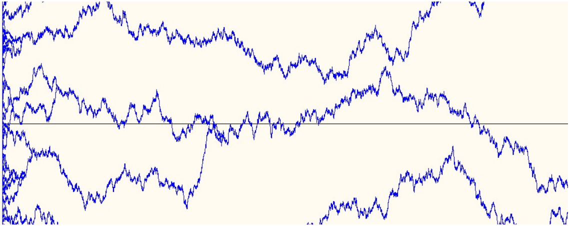
Coalescing particle system: Arratia flow

Arratia flow on $\mathbb{R}$ (R. Arratia '79)

- Brownian particles start from every point of an interval;
- they move independently and coalesce after meeting;



Mathematical description of Arratia flow



$X(u,t)$ is the position of particle at time t starting at u

1. $X(u,0) = u$;
2. $X(u,\cdot)$ is a Brownian motion in $\mathbb{R}$;
3. $X(u,t) \leq X(v,t)$, $u < v$
4. $\langle X(u,\cdot), X(v,\cdot)\rangle_t = \int_0^t \mathbb{I}_{\{X(u,s)=X(v,s)\}} ds$.

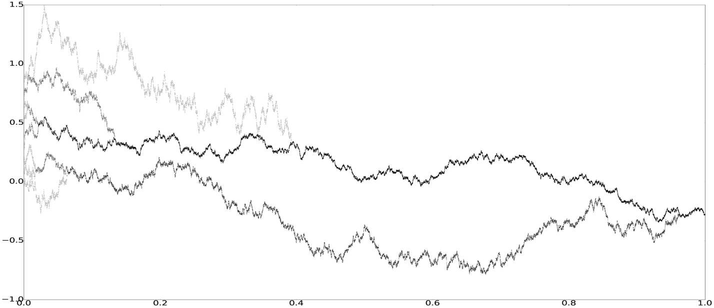
Arratia flow and its generalization

- **Arratia flow appears as scaling limit of different models**
  - true self-repelling motion (B.Tóth and W. Werner (PTRF '98))
  - isotropic stochastic flows of homeomorphisms in $\mathbb{R}$ (V. Piterbarg (Ann. Prob. '98))
  - Hastings-Levitov planer aggregation models (J. Norris, A. Turner (Comm. Math. Phys. '12)), etc. . .
- **Further investigation of the Arratia flow**
  - Properties of generated σ-algebra (B. Tsirelson (Probab. Surv. '04))
  - n-particle motion (R. Tribe, O.V. Zaboronski (EJP '04, Comm. Math. Phys. '06))
  - large deviations (A. Dorogovtsev, O. Ostapenko (Stoch. Dyn. '10)), etc. . .
- **Generalizations**
  - Brownian web (C. M. Newman et al. (Ann. Prob. '04), R. Sun, J.M Swart (MAMS, '14))
  - Coalescing non-Brownian particles (S. Evans et al. (PTRF, '13))
  - Stochastic flows of kernels (Y. Le Jan and O. Raimond (Ann. Prob. '04))

Modified Massive Arratia flow

Modified massive Arratia flow on $\mathbb{R}$ (K. (Ann. Prob. '17, EJP '17))

- Brownian particles start from points with masses;
- they move independently and coalesce after meeting;
- particles sum their masses after meeting and diffusion rate is inversely proportional to the mass.



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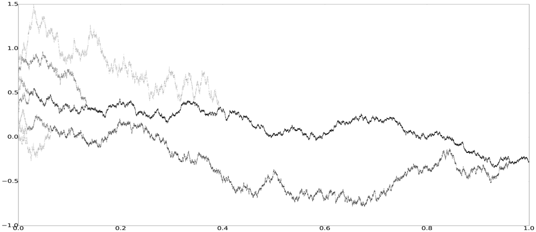
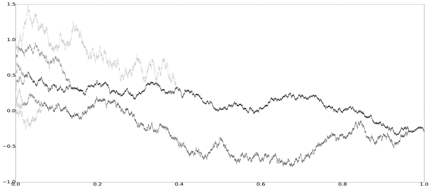


$Y(u,t)$ is the position of particle at time t labeled by $u \in (0,1)$

1. $Y(u,0) = u$;
2. $Y(u,\cdot)$ is a continuous martingale;
3. $Y(u,t) \le Y(v,t)$, $u < v$;
4. $\langle Y(u,\cdot), Y(v,\cdot)\rangle_t = \int_0^t \frac{\mathbb{I}_{\{Y(u,s)=Y(v,s)\}}}{m(u,s)} ds$,
   where $m(u,s) = \mathrm{Leb}\{v : \ Y(v,s) = Y(u,s)\}$.

Idea and problems of construction



Idea:

1. construction of a system with n particles started from $\frac{k}{n}$ with masses $\frac{1}{n}$ (diff. rate n);
2. passing to the limit as $n \to \infty$.

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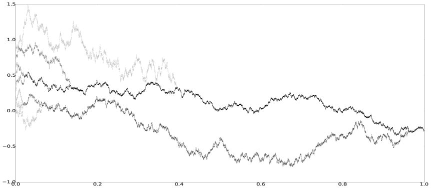

Idea:

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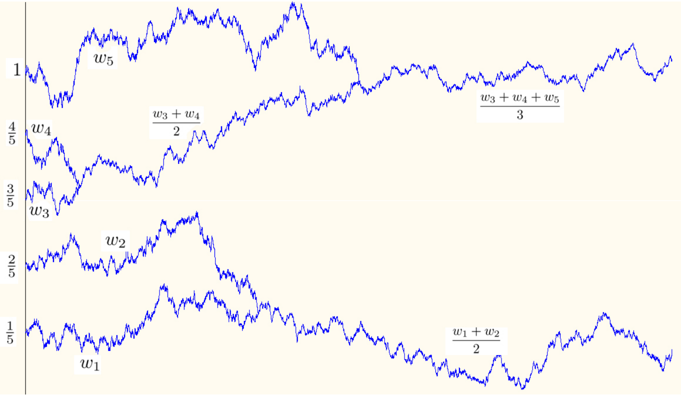
Problems:

1. adding a new particle into the system change the behavior of other particles (different from the Arratia Flow);
2. diffusion rate of particles increases
(hope: it can be compensated by coalescing of particles).

Construction of finite system

Let w_k, $k = 1, \ldots, n$, be independent Brownian motions starting from $\frac{k}{n}$ with diffusion rate $\frac{1}{n}$.



Notation: $y_k(t)$, $t \in [0, T]$, $k = 1, \ldots, n$.

Skorohod space for particle

Let $D([0,1], C[0,T])$ be a space of right continuous functions

$$[0,1] \ni u \mapsto Y(u,\cdot) \in C[0,T]$$

equipped with Skorohod distance.

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$$Y_n(u,t) := y_k(t), \quad u \in \left[\frac{k-1}{n}, \frac{k}{n}\right).$$

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Then one can show that $Y_n \in D([0,1], C[0,T])$ and

1. $Y_n(u,0) = \frac{k}{n}$ for $u \in \left[\frac{k-1}{n}, \frac{k}{n}\right)$;
2. $Y_n(u,\cdot)$ is a continuous martingale;
3. $Y_n(u,t) \leq Y_n(v,t)$, $u < v$;
4. $\langle Y_n(u,\cdot), Y_n(v,\cdot)\rangle_t = \int_0^t \frac{\mathbb{I}_{\{Y_n(u,s)=Y_n(v,s)\}}}{m_n(u,s)} ds$,
   where $m_n(u,s) = \mathrm{Leb}\{v : \ Y_n(v,s) = Y_n(u,s)\}$.

Tightness in Skorohod space

Recall that $Y_n(u,t) := y_k(t), \quad u \in \left[\frac{k-1}{n}, \frac{k}{n}\right)$.

Proposition (K. [Ann. Prob. '17])

(i) For each $u \in [0,1]$ the family $\{Y_n(u,\cdot)\}_{n\geq 1}$ is tight in $C[0,T]$.

(ii) For all $n \in \mathbb{N}$, $u \in [0,2]$, $h \in [0,u]$ and $\lambda > 0$

$$\mathbb{P}\{\|Y_n(u+h,\cdot) - Y_n(u,\cdot)\| > \lambda,\ \|Y_n(u,\cdot) - Y_n(u-h,\cdot)\| > \lambda\} \leq \frac{Ch^2}{\lambda^2}.$$

Here $y_n(u,\cdot) = y_n(1,\cdot)$, $u \in [1,2]$.

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Theorem (K. [Ann. Prob. '17])

(i) $\{Y_n\}_{n\geq 1}$ is tight in $D([0,1], C[0,T])$ and its limit point Y satisfies the martingale problem above:

1. $Y(u,0) = u$;
2. $Y(u,\cdot)$ is a continuous martingale;
3. ....

Proof of tightness of $\{Y_n(u,\cdot)\}_{n\geq 1}$

$Y_n(u,t) = y_k(t)$, $t \in [0,T]$ is a square integrable martingales with quadratic variation

$$\langle Y(u,\cdot)\rangle_t = \int_0^t \frac{1}{m_n(u,s)} ds.$$

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$$\mathbb{E}\frac{1}{m_n^{\beta}(u,t)} = \int_1^{+\infty} \mathbb{P}\left\{m_n(u,t) < \frac{1}{\tilde{r}^{1/\beta}}\right\} d\tilde{r}$$
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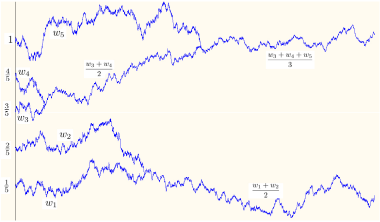
where

$$\mathbb{P}\{m_n(u,t) < r\}$$
$$\leq \mathbb{P}\{Y_n(u+r,t) - Y_n(u,t) > 0, \text{rate of diffusion of } Y_n(u,t) > 1/r\}$$
$$\leq \mathbb{P}\left\{\max_{s\in[0,t]} w\left(\frac{s}{r}\right) < r\right\} \leq \mathbb{P}\left\{\max_{s\in[0,t]} w(s) < r\sqrt{r}\right\}$$
$$\leq C_t r\sqrt{r}.$$

Equation for n particle system

w_k, $k \in [n] := \{1, \ldots, n\}$, be indep. Brownian motions starting from $\frac{k}{n}$ with diff. rate $\frac{1}{n}$.



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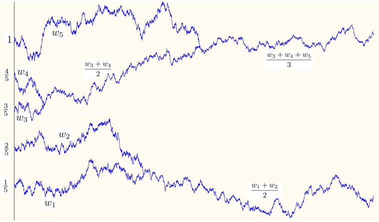

Consider $L_2[n] := \mathbb{R}^n$ as a Hilbert space of functions $x : [n] \to \mathbb{R}$ with inner product

$$(x, y)_n = \frac{1}{n} \sum_{k=1}^{n} x_k y_k.$$

$w(t) = (w_1(t), \ldots, w_n(t))$ is a cylindrical Wiener process on $\mathbb{R}^n$.

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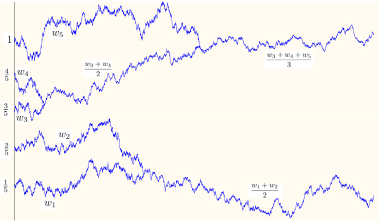

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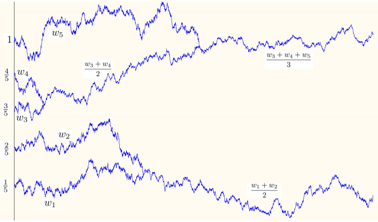

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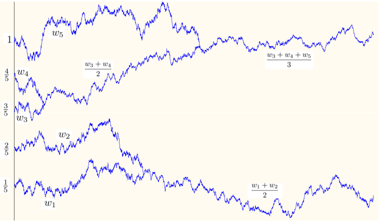

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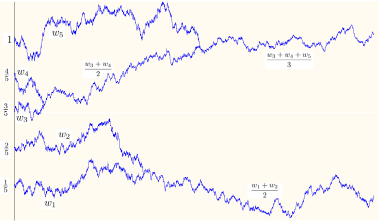

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The particle system $y(t) = (y_1(t), \ldots, y_n(t))$ takes values in $L_2^{\uparrow}[n]$ and solves the SDE

$$dy(t) = \mathrm{pr}_{\sigma(y(t))}\, dw(t).$$

Inf.-dim. equation for particle system

We similarly consider

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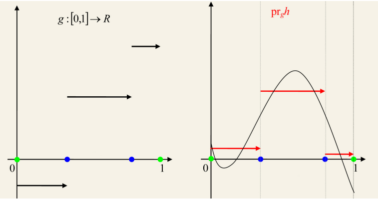

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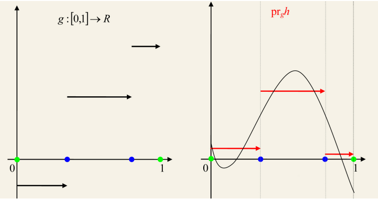
5. W_t is a cylindrical Wiener process in $L_2[0,1]$.



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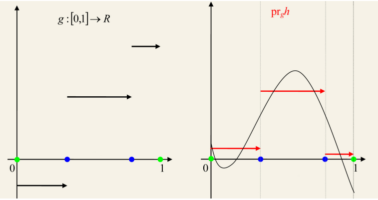

Theorem (K. [EJP '17])

The Modified Massive Arratia Flow $Y_t := Y(\cdot,t) \in L_2^{\uparrow}[0,1]$ solves the SDE

$$dY_t = \mathrm{pr}_{\sigma(Y_t)}\, dW_t,$$
$$Y_0 = id.$$

Consequences

Infinite-dimensional SDE in $L_2^{\uparrow}[0,1]$ for the Modified Massive Arratia Flow:

$$dY_t = \mathrm{pr}_{\sigma(Y_t)}\, dW_t. \qquad (1)$$

Theorem (K. [EJP '17])

For each $g \in L_{2+\varepsilon}[0,1]$ there exists a solution Y_t to the SDE (1) with $Y_0 = g$.

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For each $g \in L_{2+\varepsilon}[0,1]$ there exists a solution Y_t to the SDE (1) with $Y_0 = g$.

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Theorem [K., von Renesse (CPAM '19)]

The family of solutions to

$$dY_t^{\varepsilon} = \sqrt{\varepsilon}\, \mathrm{pr}_{\sigma(Y_t^{\varepsilon})}\, dW_t,$$
$$Y_0 = id.$$

satisfies the large deviation principle in $C([0,T], L_2^{\uparrow}[0,1]) \implies$ Connection with Wasserstein space.

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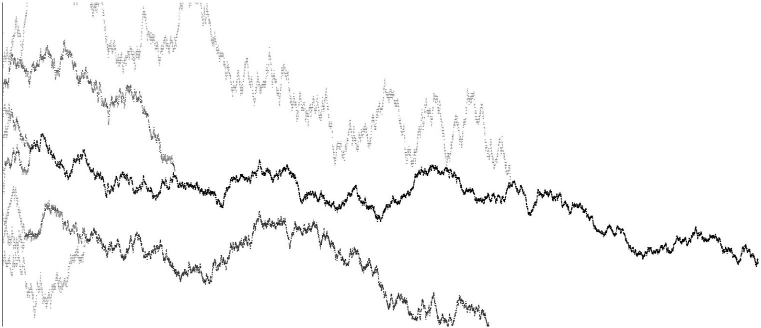
Splitting of Particles

Can we replace coalescing by another type of interaction that would lead to a reversible model?



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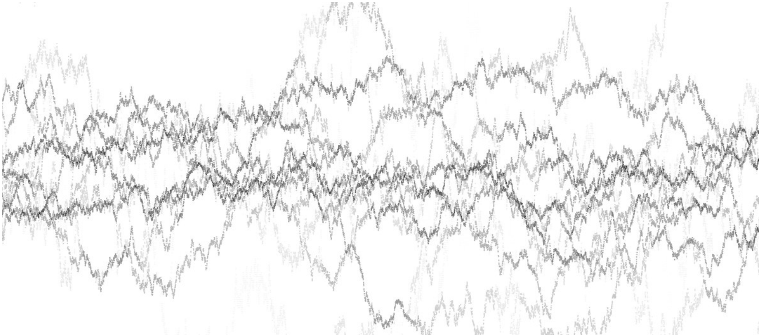
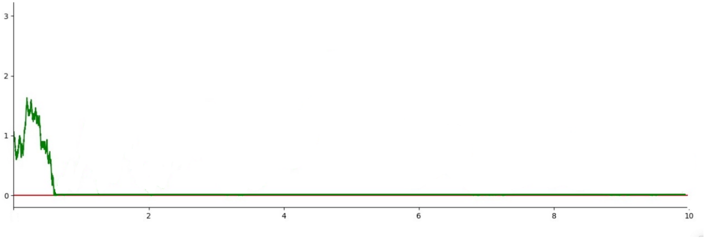


Coalescing vs. sticky-reflection (1-d case)

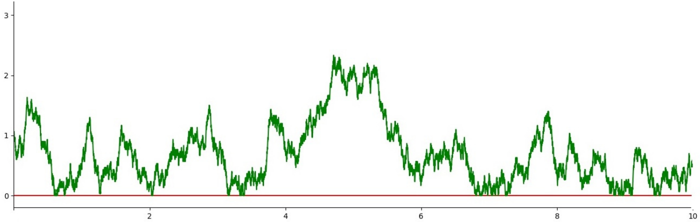
Coalescing Brownian motion on $\mathbb{R}_+$



$$dy(t) = \mathbb{I}_{\{y(t)>0\}} dw(t)$$

Coalescing vs. sticky-reflection (1-d case)

Sticky-reflected Brownian motion on $\mathbb{R}_+$

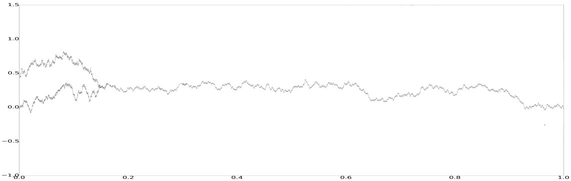


$$dy(t) = \mathbb{I}_{\{y(t)>0\}} dw(t) + \lambda \mathbb{I}_{\{y(t)=0\}} dt, \quad \lambda > 0$$

[Engelbert, Peskir '14]

Two particle model

$y_1(t) \le y_2(t)$ denote the positions of particles at time $t \ge 0$
$m_1 = m_2 = \frac{1}{2}$ particle mass at start (the total mass is always 1)



Let $w_1,\ w_2$ be two indep. Brownian motions with diffusion rate 2

$$dy_i(t) = \mathbb{I}_{\{y_1(t)<y_2(t)\}}\, dw_i(t) + \mathbb{I}_{\{y_1(t)=y_2(t)\}}\, d\,\frac{w_1(t)+w_2(t)}{2}$$

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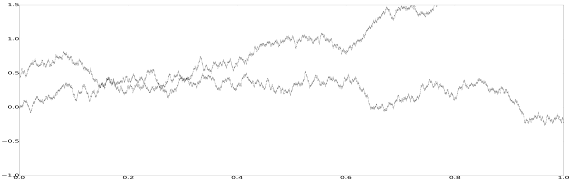

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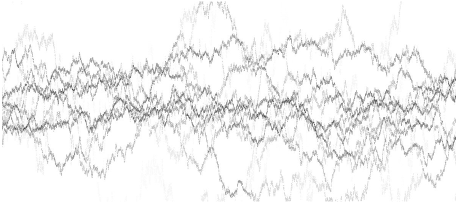
$$+\lambda_i \mathbb{I}_{\{y_1(t)=y_2(t)\}} dt, \quad \lambda_1 \leq \lambda_2$$

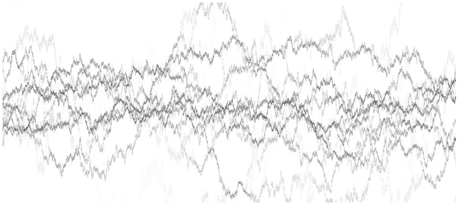
Uncountable number of particles. Diffusion term



Let $Y(u,t)$ denote the position of the particle labeld by $u \in [0,1]$ at time $t \geq 0$.

$$Y(u,t) \leq Y(v,t), \quad u \leq v$$

Uncountable number of particles. Diffusion term



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$$Y(u,t) \leq Y(v,t), \quad u \leq v$$

$$dY(u,t) = d\frac{1}{m(u,t)} \int_{\pi(u,t)} dW_t + \text{drift term}$$

where $\pi(u,t) = \{v : \ Y(u,t) = Y(v,t)\}$ and $m(u,t) = \mathrm{Leb}\{\pi(u,t)\}$

Uncountable number of particles. Drift term

$\xi(u)$ – an **interaction potential** of the particle u,

where $\xi : [0, 1] \to \mathbb{R}$, $\xi(u) \leq \xi(v)$, $u \leq v$.

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Or equivalently for $Y_t := Y(t,\cdot) \in L_2^{\uparrow}[0,1]$:

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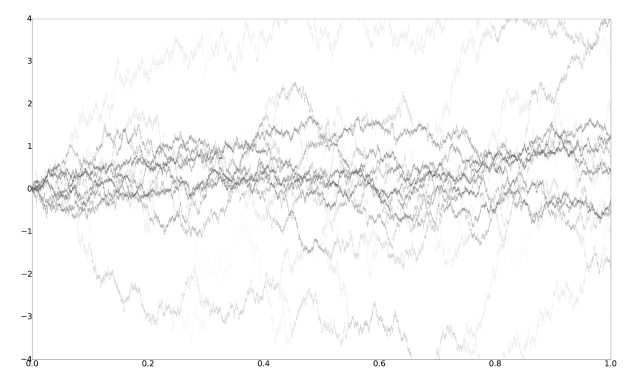
Remark: If $\xi(u) = \xi(v)$, then particles u and v coalesce.

Simulation for $\xi(u) = u$ and $Y_0(u) = 0$



$$g(u) = 0, \quad \xi(u) = u, \quad u \in [0, 1]$$

The model is similar to the Howitt-Warren flow, where particles do not change their diffusion rate. [Howitt, Warren (Ann. Probab. '09); Schertzer, Sun, Swart (Mem. Amer. Math. Soc. '14)]

Existence of particle system

Theorem (K. [Ann. Inst. H. Poincaré, '23])

Let $g, \xi : [0,1] \to \mathbb{R}$ be nondecreasing and piecewise $\frac{1}{2}+$-Hölder continuous. Then there exists a family of continuous processes $Y(u,\cdot)$, $u \in [0,1]$, such that

1. $Y(u,0) = g(u)$
2. $Y(u,\cdot) - \int_0^t \left(\xi(u) - \frac{1}{m(u,s)} \int_{\pi(u,s)} \xi(r)dr\right) ds$ – is a martingale, where $\pi(u,t) = \{v : \ Y(u,t) = Y(v,t)\}$, $m(u,s) = \mathrm{Leb}\{w : Y(w,t) = Y(u,t)\}$;
3. $Y(u,t) \leq Y(v,t)$, $u < v$;
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Uniqueness of distribution is still an important open problem.

Number of particles

Let $N(t)$ be a number of distinct particles at time t.

Theorem (K. [TSP, '20])

1. $\int_0^t \mathbb{E}N(t)dt < \infty$ a.s.

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Theorem (K. [TSP, '20])

1. $\int_0^t \mathbb{E}N(t)dt < \infty$ a.s.
2. If ξ takes infinitely many values. Then

$$\mathbb{P}\{\exists \text{ a dense set } R \subset [0, \infty) : \quad N(t) = +\infty \quad \forall t \in R\} = 1$$

Reversible Particle System

Theorem **(K., Renesse [J. Funct. Anal. '24])**

For any non-decreasing right-continuous function ξ there exist a σ-finite measure Ξ on $L_2^{\uparrow}[0,1]$ and a Markov process Y in $L_2^{\uparrow}[0,1]$ such that

- Ξ – in an invariant measure for Y.
- Y_t is a solution to

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Remark: The proof is based on the Dirichlet form approach.

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Rimannian structure on Wasserstein space

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$$= \inf\left\{\int_0^1\int_{\mathbb{R}^n}|\nabla\Phi(t,x)|^2\rho(t,x)dxdt :\ \begin{array}{r}\partial_t\rho(t,x)+\nabla\cdot(\rho(t,x)\nabla\Phi(t,x))=0,\\ \rho(0,x)=\rho^1,\ \rho(1,x)=\rho^2(x)\end{array}\right\}$$

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$$\mathrm{grad}_{\mathcal{W}}F(\rho) = -\nabla\cdot\left(\rho\nabla\frac{\delta}{\delta\rho}F(\rho)\right).$$

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$\rightsquigarrow$ Heat equation

$$\frac{\partial \mu_t}{\partial t} = \frac{\alpha}{2} \Delta \mu_t$$

is a gradient flow on the Wasserstein space:

$$\frac{\partial \mu_t}{\partial t} = -\,\mathrm{grad}_{\mathcal{W}} \left[\frac{\alpha}{2} E(\mu_t) \right]$$

[Otto (CPDE'01)]

where $E(\rho) = \int_{\mathbb{R}^d} \rho(x) \ln \rho(x) dx$

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Which measure-valued process is a "good" candidate for a Brownian motion on $\mathcal{P}_2$?

Wasse

$\rightsquigarrow$ Hea

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Short-time asymptotic of a Brownian motion

Short-time asymptotic formula for a heat kernel

$$t \ln p(t,x,y) = t \ln \left[\frac{1}{(2\pi t)^{n/2}} e^{-\frac{\|x-y\|^2}{2t}} \right] \to -\frac{\|x-y\|^2}{2}, \quad t \to 0+ .$$

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Generalizations

- Heat equation with variable coefficients in $\mathbb{R}^n$ [Varadhan (CPAM '67)]
- Smooth Riemannian manifold with Ricci curvature bound [P. Li and S.-T. Yau (Acta Math. '86)]
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Corollary

If B_t, $t \geq 0$, is a Brownian motion on a Riemannian manifold, then

$$t \ln \mathbb{P}_x \{B_t = y\} \to -\frac{d^2(x,y)}{2}, \quad t \to 0+,$$

with d being the Riemannian distance.

Short-time assymptotic of particle system

Theorem (K., Renesse [CPAM '19] and [J. Funct. Anal. '24])

Let Y be the Modified Massive Arratia Flow or Sticky-Reflected Particle System with initial mass distribution μ_0. Then the evolution of particle mass

$$\mu_t = \mu_0 \circ Y^{-1}(\cdot, t),$$

satisfies Varadhan's formula

$$t \ln \mathbb{P}\{\mu_t = \nu\} \to -\frac{\mathcal{W}_2^2(\mu_0, \nu)}{2}, \quad t \to +0.$$

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