

A Quantitative Central Limit Theorem for Simple Symmetric Exclusion Process

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Joint Kyiv-Warwick Stochastic Analysis seminar

joint work with Benjamin Gess



Universität Hamburg

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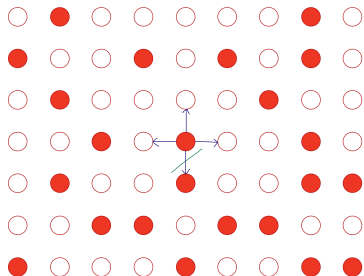
National Academy of Sciences of Ukraine
INSTITUTE OF MATHEMATICS

Simple symmetric exclusion process

On the d -dim discrete torus

$$\mathbb{T}_n^d := \left\{ \frac{k}{n} : k \in \mathbb{Z}_n^d := \{-m, \dots, m\}^d \right\} \subset \mathbb{T}^d = (\mathbb{R}/\{\mathbb{Z} - 1/2\})^d$$

for $n = 2m + 1$ we consider a **Simple Symmetric Exclusion Process (SSEP)**

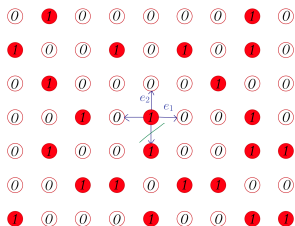


State space and generator

Particle configuration $\eta \in \{0, 1\}^{\mathbb{T}_n^d}$:

$\eta(x) = 0 \Leftrightarrow$ side x is empty

$\eta(x) = 1 \Leftrightarrow$ side x is occupied



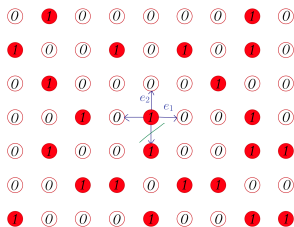
$$\eta^{x \leftrightarrow y}(z) = \begin{cases} \eta(z), & z \neq x, y, \\ \eta(y), & z = x, \\ \eta(x), & z = y, \end{cases}$$

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$$\mathcal{G}_n^{EP} F(\eta) := \frac{n^2}{2} \sum_{j=1}^d \sum_{x \in \mathbb{T}_n} [F(\eta^{x \leftrightarrow x+e_j}) - F(\eta)] \quad [\text{Kipnis, Landim '99}]$$

SSEP is already parabolically rescaled: space $\sim \frac{1}{n}$ time $\sim n^2!$

Non-equilibrium SSEP

Let $\eta_t^n, t \geq 0$, be a SSEP.

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Equilibrium SSEP:

For fixed $\rho_0 \in [0, 1]$, the measure $B(\rho_0)^{\otimes \mathbb{T}_n^d}$ is invariant for SSEP η_t^n :

If $\eta_0^n(x) \sim B(\rho_0)$, $x \in \mathbb{T}_n^d$, are independent then $\eta_t^n(x) \sim B(\rho_0)$, $x \in \mathbb{T}_n^d$, are independent.

$\rightsquigarrow \mathbb{E}\eta_t^n(x) = \rho_0$, $\text{Var} \eta_t^n = \rho_0(1 - \rho_0)$.

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What is $\mathbb{E}\eta_t^n(x)$?

Mean behavior

Set $\rho_t^n(x) := \mathbb{E}\eta_t^n(x)$.

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Then

$$d\rho_t^n(x) = \mathbb{E}\mathcal{G}_n^{EP} \eta_t^n(x) dt = \mathbb{E} \frac{n^2}{2} \sum_{j=1}^d \sum_{y \in \mathbb{T}_n} [\eta^{y \leftrightarrow y+e_j}(x) - \eta(x)] dt$$

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In particular, the empirical distribution

$$\tilde{\rho}_t^n := \frac{1}{n^d} \sum_{x \in \mathbb{T}_n^d} \rho_t(x) \delta_x,$$

solves

$$d\langle \varphi, \tilde{\rho}_t^n \rangle = \frac{1}{2} \langle \Delta_n \varphi, \tilde{\rho}_t^n \rangle dt$$

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$$d\langle \varphi, \tilde{\rho}_t^n \rangle = \frac{1}{2} \langle \Delta_n \varphi, \tilde{\rho}_t^n \rangle dt \xrightarrow{n \rightarrow \infty} d\rho_t^\infty = \frac{1}{2} \Delta \rho_t^\infty dt, \quad \rho_0^\infty = \rho_0$$

Law of large numbers

Theorem [see e.g. in Kipnis, Landim '99]

Let $\rho_0 : \mathbb{T}^d \rightarrow [0, 1]$ be an initial density profile and $\eta_0^n(x) \sim B(\rho_0(x))$ be independent. Then

$$\tilde{\eta}_t^n := \frac{1}{n^d} \sum_{x \in \mathbb{T}_n^d} \eta_t(x) \delta_x$$

converges in probability to $\rho_t^\infty(x) dx$, where $\rho_t^\infty := P_t^{HE} \rho_0$ solves

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where $\tau_j \eta(x) := \eta(x + e_j)$.

Density fluctuation field and CLT

We now consider the fluctuations of the SSEP around its mean:

$$\zeta_t^n(x) := n^{\frac{d}{2}} (\eta_t^n(x) - \rho_t^n(x)).$$

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can be expanded as follows

$$\begin{aligned} \mathcal{G}_n^{FF} f(\langle \varphi, \tilde{\zeta} \rangle) &= \frac{1}{2} f'(\langle \varphi, \tilde{\zeta} \rangle) \langle \Delta_n \varphi, \tilde{\zeta} \rangle + \frac{n^d}{4n^d} f''(\langle \varphi, \tilde{\zeta} \rangle) \langle |\partial_{n,j} \varphi|^2, \tau_j \tilde{\eta} + \tilde{\eta} - 2\widetilde{\eta \tau_j \eta} \rangle \\ &\quad + O\left(1/n^{\frac{d}{2}+1}\right) \end{aligned}$$

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$$\begin{aligned} d\langle \varphi, \tilde{\zeta}_t^n \rangle &= \frac{1}{2} \langle \Delta_n \varphi, \tilde{\zeta}_t^n \rangle dt + \text{mart.} \rightarrow d\langle \varphi, \zeta_t^\infty \rangle = \frac{1}{2} \langle \Delta \varphi, \zeta_t^\infty \rangle dt + \text{mart.} \\ d\langle \text{mart.} \rangle_t &= \frac{1}{2} \left\langle |\partial_{n,j} \varphi|^2, \tau_j \tilde{\eta}_t^n + \tilde{\eta}_t^n - 2\widetilde{\eta_t^n \tau_j \eta_t^n} \right\rangle dt \rightarrow d\langle \text{mart.} \rangle = \langle \Delta \varphi, \rho_t^\infty - \rho_t^\infty \rho_t^\infty \rangle dt \end{aligned}$$

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Central limit theorem

Theorem 2 [Galves, Kipnis, Spohn; Ravishankar '90]

Let the initial density profile ρ_0 be smooth. Then the density fluctuation field

$$\tilde{\zeta}_t^n := \frac{1}{n^d} \sum_{x \in \mathbb{T}_n^d} \zeta_t(x) \delta_x$$

converges in $D([0, T], \mathcal{D}')$ to the generalized Ornstein-Uhlenbeck process that solves the linear SPDE

$$d\zeta_t^\infty = \frac{1}{2} \Delta \zeta_t^\infty dt + \nabla \cdot \left(\sqrt{\rho_t^\infty (1 - \rho_t^\infty)} dW_t \right)$$

with the centered Gaussian initial condition such that

$$\mathbb{E} [\langle \zeta_0^\infty, \varphi \rangle^2] = \langle \rho_0 (1 - \rho_0) \varphi, \varphi \rangle$$

Idea of proof

- Consider the semimartingale $\langle \varphi, \tilde{\zeta}_t \rangle$ and its quadratic variation.

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$$\eta_t^n \tau_j \eta_t^n = \rho_t^n \tau_j \rho_t^n + \frac{1}{n^{d/2}} (\rho_t^n \tau_j \zeta_t^n + \zeta_t^n \tau_j \rho_t^n) + \frac{1}{n^d} \zeta_t^n \tau_j \zeta_t^n,$$

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- Control of $\mathbb{E} \left[\frac{1}{n^d} \zeta_t^n(x) \tau_j \zeta_t^n(x) \right] = \mathbb{E} [(\eta_t^n(x) - \rho_t^n(x)) (\eta_t^n(x + e_j) - \rho_t^n(x + e_j))] \rightarrow 0$
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[Ferrari et al. '91]
- Use of tightness argument and the Holley-Stroock theory
[Holley, Stroock '79]

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$$\tilde{\zeta}_t^n - \zeta_t^\infty = O(?)$$

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- [Kolokoltsov '10] Central limit theorem for the Smoluchovski coagulation model (mean field limit, non-local Smoluchowski's coagulation equation)
- ...

Main result

Theorem 3 [Gess, K. '24]

Let

- the initial density profile $\rho_0 : \mathbb{T}^d \rightarrow [0, 1]$ be smooth enough,
- η_t^n be SSEP with $\eta_0^n(x) \sim B(\rho_0(x))$ and independent,
- $\rho_t^n = \mathbb{E}\eta_t^n$, $\zeta_t^n = n^{d/2}(\eta_t^n - \rho_t^n)$
- ζ_t^∞ solves $d\zeta_t^\infty = \frac{1}{2}\Delta\zeta_t^\infty dt + \nabla \cdot \left(\sqrt{\rho_t^\infty(1-\rho_t^\infty)} dW_t \right)$ with the centered Gaussian initial condition with $\mathbb{E} [\langle \zeta_0^\infty, \varphi \rangle^2] = \langle \rho_0(1-\rho_0)\varphi, \varphi \rangle$

Then for large enough $l \in \mathbb{N}$

$$\sup_{t \in [0, T]} \left| \mathbb{E} f(\langle \vec{\varphi}, \tilde{\zeta}_t^n \rangle) - \mathbb{E} f(\langle \vec{\varphi}, \zeta_t^\infty \rangle) \right| \leq \frac{C}{n^{\frac{d}{2} \wedge 1}} \|f\|_{C_b^3} \|\vec{\varphi}\|_{C^l}$$

for all $n \geq 1$, $f \in C_b^3(\mathbb{R}^m)$ and $\vec{\varphi} \in (C^l(\mathbb{T}^d))^m$.

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Theorem 3 [Gess, K. '24]

Let

- the initial density profile $\rho_0 : \mathbb{T}^d \rightarrow [0, 1]$ be smooth enough,
- η_t^n be SSEP with $\eta_0^n(x) \sim B(\rho_0(x))$ and independent,
- $\rho_t^n = \mathbb{E}\eta_t^n$, $\zeta_t^n = n^{d/2}(\eta_t^n - \rho_t^n)$
- ζ_t^∞ solves $d\zeta_t^\infty = \frac{1}{2}\Delta\zeta_t^\infty dt + \nabla \cdot \left(\sqrt{\rho_t^\infty(1 - \rho_t^\infty)} dW_t \right)$ with the centered Gaussian initial condition with $\mathbb{E} [\langle \zeta_0^\infty, \varphi \rangle^2] = \langle \rho_0(1 - \rho_0)\varphi, \varphi \rangle$

Then for large enough $l \in \mathbb{N}$

$$\sup_{t \in [0, T]} \left| \mathbb{E} f(\langle \vec{\varphi}, \zeta_t^n \rangle) - \mathbb{E} f(\langle \vec{\varphi}, \zeta_t^\infty \rangle) \right| \leq \frac{C}{n^{\frac{d}{2} \wedge 1}} \|f\|_{C^3} \|\vec{\varphi}\|_{C^l}$$

for all $n \geq 1$, $f \in C_b^3(\mathbb{R}^m)$ and $\vec{\varphi} \in (C^l(\mathbb{T}^d))^m$.

The rate $\frac{1}{n^{\frac{d}{2} \wedge 1}}$ is optimal: $\frac{1}{n}$ – lattice discretization error, $\frac{1}{n^{\frac{d}{2}}}$ – particle approximation error

Main tool

Idea of proof: Compare two (time-homogeneous) Markov processes X_t, Y_t taking values in the **same state space** and $X_0 = Y_0 = x$ using

$$\begin{aligned}\mathbb{E}F(X_t) - \mathbb{E}F(Y_t) &= \int_0^t P_s^X (\mathcal{G}^X - \mathcal{G}^Y) P_{t-s}^Y F(x) ds, \\ &= \int_0^t \mathbb{E} [(\mathcal{G}^X - \mathcal{G}^Y) P_{t-s}^Y F(X_s)] ds,\end{aligned}$$

[see e.g. Ethier, Kurtz '86]

Splitting of the problem

Recall

$$\mathbb{E}F(X_t) - \mathbb{E}F(Y_t) = \int_0^t \mathbb{E} \left[(G^X - G^Y) P_{t-s}^Y F(X_s) \right] ds,$$

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where $X_0 = Y_0 = x$.

We consider the Markov processes:

- particle means and fluctuation field: $(\tilde{\rho}_t^n, \tilde{\zeta}_t^n)$
- solution to heat equation and generalized OU process $(\rho_t^\infty, \zeta_t^\infty)$.

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The processes starts from different initial conditions!

We will compare:

- $(\tilde{\rho}_t^n, \tilde{\zeta}_t^n)$ and $(\rho_t^{\infty,n}, \zeta_t^{\infty,n})$ [**comparison of dynamics**]
where the generalized OU process $(\rho_t^{\infty,n}, \zeta_t^{\infty,n})$ started from $(\tilde{\rho}_0^n, \tilde{\zeta}_0^n)$;
- $(\rho_t^{\infty,n}, \zeta_t^{\infty,n})$ and $(\rho_t^\infty, \zeta_t^\infty)$ [**comparison of initial conditions**]
(both are defined by the same equation).

Generators

We start from the formal computation for cylindrical functions:

$$F(\tilde{\rho}, \tilde{\zeta}) := f(\langle \psi, \tilde{\rho} \rangle, \langle \varphi, \tilde{\zeta} \rangle)$$

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where $\tilde{\zeta} = n^{d/2}(\tilde{\eta} - \tilde{\rho})$.

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where $\tilde{\zeta} = n^{d/2}(\tilde{\eta} - \tilde{\rho})$. Moreover,

$$\begin{aligned} \mathcal{G}^{OU} F(\rho^\infty, \zeta^\infty) &= \frac{1}{2} \partial_1 f \langle \Delta \varphi, \rho^\infty \rangle + \frac{1}{2} \partial_2 f \langle \Delta \varphi, \zeta^\infty \rangle \\ &\quad + \frac{1}{2} \partial_2^2 f \langle |\partial_j \varphi|^2, \rho^\infty - (\rho^\infty)^2 \rangle. \end{aligned}$$

The difficulties

- Generators have to be compared on $U := P_{t-s}^{OU} f(\langle \psi, \cdot \rangle, \langle \varphi, \cdot \rangle)$, that is **not cylindrical function**.
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- We compare the generators on particle configurations $\tilde{\rho}_t^n, \tilde{\zeta}_t^n$, that are empirical distributions. But **U is not differentiable at $\tilde{\rho}_t^n$** because of the term $\sqrt{\rho(1-\rho)}$ in the SPDE
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- Control of $\mathbb{E} \left[f(\langle \varphi, \tilde{\zeta}_t^n \rangle) \left\langle \widetilde{\zeta_t^n \tau \zeta_t^n}, |\partial_{n,j} \varphi|^2 \right\rangle \right]$ via $\mathbb{E} \prod_{i=1}^m (\eta_t^n(x_i) - \rho_t^n(x_i))$ **does not give the optimal rate**.

Discrete and continuous Fourier transform

Replace $\tilde{\rho} = \frac{1}{n^d} \sum_{x \in \mathbb{T}_n^d} \rho(x) \delta_x$ and $\tilde{\zeta} = \frac{1}{n^d} \sum_{x \in \mathbb{T}_n^d} \zeta(x) \delta_x$ by a smooth interpolation.

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- Let $L_2(\mathbb{T}_n^d)$ be the Hilbert space of all functions on \mathbb{T}_n^d with inner product

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- $\varsigma_k(x) = e^{2\pi i k \cdot x}$, $k \in \mathbb{Z}^d$, $x \in \mathbb{T}^d \supset \mathbb{T}_n^d$
 - basis vectors on $L_2(\mathbb{T}_n^d)$ and $L_2(\mathbb{T}^d)$, and
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$$L_2(\mathbb{T}_n^d) \ni \rho = \sum_{k \in \mathbb{Z}_n^d} \langle \rho, \varsigma_k \rangle_n \varsigma_k \quad \text{on } \mathbb{T}_n^d, \quad L_2(\mathbb{T}^d) \ni g = \sum_{k \in \mathbb{Z}^d} \langle g, \varsigma_k \rangle \varsigma_k \quad \text{on } \mathbb{T}^d$$

New (smooth) lifting of discrete space

For functions $\rho \in L_2(\mathbb{T}_n^d)$ and $\varphi \in L_2(\mathbb{T}^d)$ define

$$\text{ex}_n \rho := \sum_{k \in \mathbb{Z}_n^d} \langle \rho, \varsigma_k \rangle_n \varsigma_k \quad \text{on } \mathbb{T}^d, \quad \text{pr}_n \varphi := \sum_{k \in \mathbb{Z}_n^d} \langle \varphi, \varsigma_k \rangle \varsigma_k \quad \text{on } \mathbb{T}^d$$

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Basic properties of $\text{ex}_n f$ and $\text{pr}_n g$

- $\text{ex}_n \rho = \rho$ on \mathbb{T}_n^d and $\text{ex}_n \rho \in C^\infty(\mathbb{T}^d)$
- $\text{pr}_n \varphi$ is well defined on \mathbb{T}_n^d for each $\varphi \in H_J$ for $H_J := \{ \varphi : \|\varphi\|_{H_J}^2 := \sum_{k \in \mathbb{Z}^d} (1 + |k|^2)^J |\langle \varphi, \varsigma_k \rangle|^2 \}$, $J \in \mathbb{R}$.
- $\langle \rho_1, \rho_2 \rangle_n = \langle \text{ex}_n \rho_1, \text{ex}_n \rho_2 \rangle$ and $\langle \rho, \text{pr}_n g \rangle_n = \langle \text{ex}_n \rho, g \rangle$
- $\|\text{pr}_n g - g\|_{H_J} \leq \frac{1}{n} \|g\|_{H_{J+1}}$, $\|\text{ex}_n \varphi - \varphi\|_{H_J} \leq \frac{C}{n} \|\varphi\|_{C^{J+2+\frac{d}{2}}}, \dots$

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- $\langle \rho_1, \rho_2 \rangle_n = \langle \text{ex}_n \rho_1, \text{ex}_n \rho_2 \rangle$ and $\langle \rho, \text{pr}_n g \rangle_n = \langle \text{ex}_n \rho, g \rangle$
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$$\begin{aligned} \langle \varphi, \tilde{\rho} \rangle &= \frac{1}{(2n+1)^d} \sum_{x \in \mathbb{T}_n^d} \varphi(x) \rho(x) = \langle \varphi, \rho \rangle_n \\ &= \langle \text{pr}_n \varphi, \rho \rangle_n + O(1/n) = \langle \varphi, \text{ex}_n \rho \rangle + O(1/n) \end{aligned}$$

Comparison of generators for smooth interpolation

Now for $F(\text{ex}_n\rho, \text{ex}_n\zeta) := f(\langle\psi, \text{ex}_n\rho\rangle, \langle\varphi, \text{ex}_n\zeta\rangle) = f(\langle\text{pr}_n\psi, \rho\rangle_n, \langle\text{pr}_n\varphi, \zeta\rangle_n)$,

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$$|\mathcal{G}_n^{FF} F - \mathcal{G}_n^{OU} F| \lesssim \frac{1}{n^{d/2}} \left| \langle \text{ex}_n\zeta, \text{ex}_n (\tau_j\rho |\partial_{n,j}\varphi|^2) \rangle \right| + \left\langle \frac{1}{n^d} \text{ex}_n (\zeta\tau_j\zeta), \text{ex}_n \partial_2^2 f (\partial_{n,j}\text{pr}_n\varphi)^2 \right\rangle^2 +$$

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We now need only to control:

- The expectations:

$$\mathbb{E} \left| \langle \text{ex}_n \zeta_t^n, \text{ex}_n (\tau_j \rho_t^n |\partial_{n,j} \varphi|^2) \rangle \right| \lesssim \|\text{ex}_n \rho_t^n\|_{C^J} \mathbb{E} \|\text{ex}_n \zeta_t^n\|_{H_{-1}}$$

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- The expectations:

$$\mathbb{E} \left| \langle \text{ex}_n \zeta_t^n, \text{ex}_n (\tau_j \rho_t^n |\partial_{n,j} \varphi|^2) \rangle \right| \lesssim \|\text{ex}_n \rho_t^n\|_{C^J} \mathbb{E} \|\text{ex}_n \zeta_t^n\|_{H_{-l}}$$

- Using the Fourier analysis, the term $\mathbb{E} \left\langle \frac{1}{n^d} \text{ex}_n (\zeta_t^n \tau_j \zeta_t^n), \text{ex}_n \partial_2^2 f(\dots)^2 \right\rangle$ can be controlled via

$$\mathbb{E} \prod_{i=1}^4 (\eta_t^n(x_i) - \rho_t^n(x_i)) \lesssim \frac{1}{n}$$

Comparison of generators for smooth interpolation

Now for $F(\text{ex}_n \rho, \text{ex}_n \zeta) := f(\langle \psi, \text{ex}_n \rho \rangle, \langle \varphi, \text{ex}_n \zeta \rangle) = f(\langle \text{pr}_n \psi, \rho \rangle_n, \langle \text{pr}_n \varphi, \zeta \rangle_n)$, we get

$$\begin{aligned} \mathcal{G}_n^{FF} F(\text{ex}_n \rho, \text{ex}_n \zeta) &= \frac{1}{2} \partial_1 f \langle \text{ex}_n \Delta_n \text{pr}_n \psi, \text{ex}_n \rho \rangle + \frac{1}{2} \partial_2 f \langle \text{ex}_n \Delta_n \text{pr}_n \varphi, \text{ex}_n \zeta \rangle \\ &\quad + \frac{1}{2} \partial_2^2 f \sum_{j=1}^d \langle \text{ex}_n (\partial_{n,j} \text{pr}_n \varphi)^2, \text{ex}_n (\tau_j \eta + \eta - 2\eta \tau_j \eta) \rangle + O\left(\frac{1}{n^{d/2+1}}\right) \end{aligned}$$

$$|\mathcal{G}_n^{FF} F - \mathcal{G}_n^{OU} F| \lesssim \frac{1}{n^{d/2}} \left| \langle \text{ex}_n \zeta, \text{ex}_n (\tau_j \rho |\partial_{n,j} \varphi|^2) \rangle \right| + \left\langle \frac{1}{n^d} \text{ex}_n (\zeta \tau_j \zeta), \text{ex}_n \partial_2^2 f (\partial_{n,j} \text{pr}_n \varphi)^2 \right\rangle^2 +$$

We now need only to control:

- The expectations:

$$\mathbb{E} \left| \langle \text{ex}_n \zeta_t^n, \text{ex}_n (\tau_j \rho_t^n |\partial_{n,j} \varphi|^2) \rangle \right| \lesssim \|\text{ex}_n \rho_t^n\|_{C^J} \mathbb{E} \|\text{ex}_n \zeta_t^n\|_{H_{-J}}$$

- Using the Fourier analysis, the term $\mathbb{E} \left\langle \frac{1}{n^d} \text{ex}_n (\zeta_t^n \tau_j \zeta_t^n), \text{ex}_n \partial_2^2 f(\dots)^2 \right\rangle$ can be controlled via

$$\mathbb{E} \prod_{i=1}^4 (\eta_t^n(x_i) - \rho_t^n(x_i)) \lesssim \frac{1}{n}$$

- We can compare generators on Frechet diff. functions on $H_J \times H_{-J}$

Differentiability of $P_t^{OU} F(\text{ex}_n \rho, \text{ex}_n \zeta)$

A solution to

$$d\rho_t^\infty = \frac{1}{2} \Delta \rho_t^\infty dt$$

$$d\zeta_t^\infty = \frac{1}{2} \Delta \zeta_t^\infty dt + \nabla \cdot \left(\sqrt{\rho_t^\infty (1 - \rho_t^\infty)} dW_t \right)$$

exists for all $\rho_0^\infty \in L_2(\mathbb{T}^d; [0, 1])$ and $\zeta_0^\infty \in H_{-l}$ for $l > \frac{d}{2} + 1$.

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For $F \in C(H_J \times H_{-l})$ (e.g. $F = f(\langle \psi, \cdot \rangle, \langle \varphi, \cdot \rangle)$) define $U_t(\rho_0^\infty, \zeta_0^\infty) := \mathbb{E} F(\rho_t^\infty, \zeta_t^\infty)$

Differentiability of $P_t^{OU} F(\text{ex}_n \rho, \text{ex}_n \zeta)$

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For $F \in C(H_J \times H_{-l})$ (e.g. $F = f(\langle \psi, \cdot \rangle, \langle \varphi, \cdot \rangle)$) define $U_t(\rho_0^\infty, \zeta_0^\infty) := \mathbb{E} F(\rho_t^\infty, \zeta_t^\infty)$

Proposition 1 [Gess, K. '24]

Let $l > \frac{d}{2} + 1$ and $F \in C_b^{2,4}(H_{-l})$. Then $U_t(\rho_0^\infty, \zeta_0^\infty) = \mathbb{E} F(\zeta_t^\infty) \in C_b^{1,3}(H_J \times H_{-l})$ for $J > \frac{d}{2}$. Moreover,

$$D_1 U_t(\rho_0^\infty, \zeta_0^\infty)[h] = \frac{1}{2} \mathbb{E} [D^2 F(\zeta_t^\infty) : DV_t(\rho_0^\infty)[h]]$$

with

$$V_t(\rho_0^\infty)(\varphi, \psi) = \text{Cov}(\langle \varphi, \zeta_t^\infty \rangle, \langle \psi, \zeta_t^\infty \rangle)$$

$$= \frac{1}{2} \int_0^t \langle \nabla P_{t-s}^{HE} \varphi \cdot \nabla P_{t-s}^{HE} \psi, \rho_s^\infty (1 - \rho_s^\infty) \rangle ds$$

Comparison of dynamics

Recall that (ρ_t^n, ζ_t^n) is the mean process together with the fluctuation field of SSEP.

Comparison of dynamics

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$(\rho_t^{\infty, n}, \zeta_t^{\infty, n})$ is a solution to

$$d\rho_t^\infty = \frac{1}{2} \Delta \rho_t^\infty dt$$

$$d\zeta_t^\infty = \frac{1}{2} \Delta \zeta_t^\infty dt + \nabla \cdot \left(\sqrt{\rho_t^\infty (1 - \rho_t^\infty)} dW_t \right)$$

started from $(ex_n \rho_0^n, ex_n \zeta_0^n)$.

Comparison of dynamics

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$(\rho_t^{\infty, n}, \zeta_t^{\infty, n})$ is a solution to

$$\begin{aligned} d\rho_t^\infty &= \frac{1}{2} \Delta \rho_t^\infty dt \\ d\zeta_t^\infty &= \frac{1}{2} \Delta \zeta_t^\infty dt + \nabla \cdot \left(\sqrt{\rho_t^\infty (1 - \rho_t^\infty)} dW_t \right) \end{aligned}$$

started from $(\text{ex}_n \rho_0^n, \text{ex}_n \zeta_0^n)$.

Then for each I, J large enough and $F \in C_b(H_J \times H_{-I})$

$$\begin{aligned} |\mathbb{E} F(\text{ex}_n \rho_t^n, \text{ex}_n \zeta_t^n) - \mathbb{E} F(\rho_t^{\infty, n}, \zeta_t^{\infty, n})| &\leq \int_0^t \left| \mathbb{E} \left[(\mathcal{G}^{FF} - \mathcal{G}^{OU}) P_{t-s}^{OU} F(\text{ex}_n \rho_s^n, \text{ex}_n \zeta_s^n) \right] \right| ds \\ &\lesssim \|F\|_{C^{1,3}} \int_0^t \left(\frac{1}{n} + \frac{1}{n^{d/2}} \|\text{ex}_n \rho_s^n\|_{C^J} \mathbb{E} \|\text{ex}_n \zeta_s^n\|_{H_{-I}} + \mathbb{E} \langle \text{ex}_n (\zeta_s^n \tau_j \zeta_s^n), \dots \rangle^2 + \dots \right) ds \end{aligned}$$

Berry-Esseen bound for the initial fluctuations

- It remains only to compare

$$\mathbb{E}F(\rho_t^{\infty,n}, \zeta_t^{\infty,n}) - \mathbb{E}F(\rho_t^\infty, \zeta_t^\infty) = P_t^{OU}F(\text{ex}_n\rho_0^n, \text{ex}_n\zeta_0^n) - P_t^{OU}F(\rho_0, \zeta_0)$$

where ρ_t^∞ started from the initial profile ρ_0 and ζ_t^∞ started from the centered Gaussian distribution with

$$\mathbb{E}\langle \zeta_0, \varphi \rangle^2 = \langle \rho_0(1 - \rho_0)\varphi, \varphi \rangle.$$

- It is enough to compare only

$$\mathbb{E}F(\text{ex}_n\zeta_0^n) - \mathbb{E}F(\text{pr}_n\zeta_0),$$

for $F \in C^3(H_{-l})$, where

$$\text{ex}_n\zeta_0^n := \sum_{k \in \mathbb{Z}_n^d} \langle \zeta_0^n, s_k \rangle_n s_k, \quad \text{pr}_n\zeta_0 := \sum_{k \in \mathbb{Z}_n^d} \langle \zeta_0, s_k \rangle s_k$$

- Is enough to compare for $f \in C^3(\mathbb{R}^{\mathbb{Z}_n^d})$

$$\mathbb{E}f\left(\left((1 + |k|^2)^{-l/2} \langle \zeta_0^n, s_k \rangle_n\right)_{k \in \mathbb{Z}_n^d}\right) - \mathbb{E}f\left(\left((1 + |k|^2)^{-l/2} \langle \zeta_0, s_k \rangle\right)_{k \in \mathbb{Z}_n^d}\right).$$

- Apply multidimensional Berry-Esseen theorem [Meckes '09]

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Thank you!