

A Quantitative Central Limit Theorem for Simple Symmetric Exclusion Process

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DER FORSCHUNG | DER LEHRE | DER BILDUNG



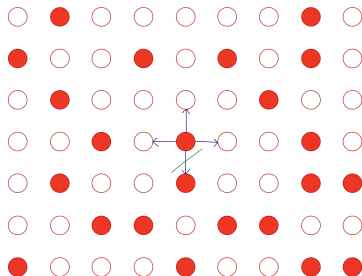
National Academy of Sciences of Ukraine
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Simple symmetric exclusion process

On the d -dim discrete torus

$$\mathbb{T}_n^d := \left\{ \frac{k}{n} : k \in \mathbb{Z}_n^d := \{0, \dots, n-1\}^d \right\} \subset \mathbb{T}^d = (\mathbb{R}/\mathbb{Z})^d$$

we consider a **Simple Symmetric Exclusion Process (SSEP)**

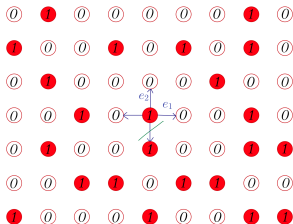


State space and generator

Particle configuration $\eta \in \{0, 1\}^{\mathbb{T}_n^d}$:

$\eta(x) = 0 \Leftrightarrow$ side x is empty

$\eta(x) = 1 \Leftrightarrow$ side x is occupied



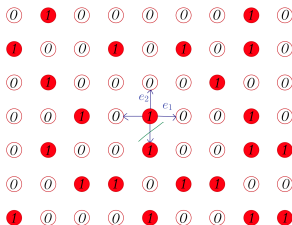
$$\eta^{x \leftrightarrow y}(z) = \begin{cases} \eta(z), & z \neq x, y, \\ \eta(y), & z = x, \\ \eta(x), & z = y, \end{cases}$$

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$$\mathcal{G}_n^{EP} F(\eta) := \frac{n^2}{2} \sum_{j=1}^d \sum_{x \in \mathbb{T}_n} [F(\eta^{x \leftrightarrow x+e_j}) - F(\eta)] \quad [\text{Kipnis, Landim '99}]$$

SSEP is already parabolically rescaled: space $\sim \frac{1}{n}$ time $\sim n^2$!

Non-equilibrium SSEP

Let η_t^n , $t \geq 0$, be a SSEP and $\rho_0 : \mathbb{T}^d \rightarrow [0, 1]$ be an initial profile.

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The process $\rho_t(x) := \mathbb{E}\eta_t^n(x)$ solves the discrete stochastic Heat equation

$$d\rho_t^n(x) = \frac{1}{2} \Delta_n \rho_t^n(x) dt$$

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Thus,

$$\tilde{\rho}_t^n := \frac{1}{n^d} \sum_{x \in \mathbb{T}_n^d} \rho_t^n(x) \delta_x \rightarrow \rho_t^\infty dx,$$

where $\rho_t^\infty := P_t^{HE} \rho_0$ solves

$$d\rho_t^\infty = \frac{1}{2} \Delta \rho_t^\infty dt, \quad \rho_0^\infty = \rho_0.$$

Law of large numbers

Theorem [see e.g. in Kipnis, Landim '99]

Let $\rho_0 : \mathbb{T}^d \rightarrow [0, 1]$ be an initial density profile and $\eta_0^n(x) \sim B(\rho_0(x))$ be independent. Then

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$$\begin{aligned} \mathcal{G}_n^{EP} f(\langle \varphi, \tilde{\eta} \rangle) &:= \frac{n^2}{2} \sum_{j=1}^d \sum_{x \in \mathbb{T}_n} \left[f(\langle \varphi, \tilde{\eta}^{x \leftrightarrow x+e_j} \rangle) - f(\langle \varphi, \tilde{\eta} \rangle) \right] \\ &= \frac{1}{2} f'(\langle \varphi, \tilde{\eta} \rangle) \langle \Delta_n \varphi, \tilde{\eta} \rangle + \frac{1}{4n^d} f''(\langle \varphi, \tilde{\eta} \rangle) \left\langle \left| \partial_{n,j} \varphi \right|^2, \tau_j \tilde{\eta} + \tilde{\eta} - 2\widetilde{\eta \tau_j \eta} \right\rangle + \dots, \end{aligned}$$

where $\tau_j \eta(x) := \eta(x + e_j)$.

Density fluctuation field and CLT

We now consider the fluctuations of the SSEP around its mean:

$$\zeta_t^n(x) := n^{\frac{d}{2}} (\eta_t^n(x) - \rho_t^n(x)).$$

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$$\begin{aligned} \mathcal{G}_n^{FF} f(\langle \varphi, \tilde{\zeta} \rangle) &= \frac{1}{2} f'(\langle \varphi, \tilde{\zeta} \rangle) \langle \Delta_n \varphi, \tilde{\zeta} \rangle + \frac{n^d}{4n^d} f''(\langle \varphi, \tilde{\zeta} \rangle) \langle |\partial_{n,j} \varphi|^2, \tau_j \tilde{\eta} + \tilde{\eta} - 2\widetilde{\eta \tau_j \eta} \rangle \\ &\quad + O\left(1/n^{\frac{d}{2}+1}\right) \end{aligned}$$

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Central limit theorem

Theorem 2 [Galves, Kipnis, Spohn; Ravishankar '90]

Let the initial density profile ρ_0 be smooth. Then the density fluctuation field

$$\tilde{\zeta}_t^n := \frac{1}{n^d} \sum_{x \in \mathbb{T}_n^d} \zeta_t(x) \delta_x$$

converges in $D([0, T], \mathcal{D}')$ to the generalized Ornstein-Uhlenbeck process that solves the linear SPDE

$$d\zeta_t^\infty = \frac{1}{2} \Delta \zeta_t^\infty dt + \nabla \cdot \left(\sqrt{\rho_t^\infty (1 - \rho_t^\infty)} dW_t \right)$$

with the centered Gaussian initial condition such that

$$\mathbb{E} [\langle \zeta_0^\infty, \varphi \rangle^2] = \langle \rho_0 (1 - \rho_0) \varphi, \varphi \rangle$$

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(mean field limit)
- [Kolokoltsov '10] Central limit theorem for the Smoluchovski coagulation model
(mean field limit, non-local Smoluchowski's coagulation equation)
- ...

Main result

Theorem 3 [Gess, K. '24]

Let

- the initial density profile $\rho_0 : \mathbb{T}^d \rightarrow [0, 1]$ be smooth enough,
- η_t^n be SSEP with $\eta_0^n(x) \sim B(\rho_0(x))$ and independent,
- $\rho_t^n = \mathbb{E}\eta_t^n$, $\zeta_t^n = n^{d/2}(\eta_t^n - \rho_t^n)$
- ζ_t^∞ solves $d\zeta_t^\infty = \frac{1}{2}\Delta\zeta_t^\infty dt + \nabla \cdot \left(\sqrt{\rho_t^\infty(1-\rho_t^\infty)} dW_t \right)$ with the centered Gaussian initial condition with $\mathbb{E}[\langle \zeta_0^\infty, \varphi \rangle^2] = \langle \rho_0(1-\rho_0)\varphi, \varphi \rangle$

Then

$$\sup_{t \in [0, T]} \left| \mathbb{E}f(\langle \vec{\varphi}, \tilde{\zeta}_t^n \rangle) - \mathbb{E}f(\langle \vec{\varphi}, \zeta_t^\infty \rangle) \right| \leq \frac{C}{n^{\frac{d}{2} \wedge 1}} \|f\|_{C^3} \|\vec{\varphi}\|_{C^I}$$

for all $n \geq 1$, $f \in C_b^3(\mathbb{R}^m)$ and $\vec{\varphi} \in (C^I(\mathbb{T}^d))^m$, where I is large enough.

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for all $n \geq 1$, $f \in C_b^3(\mathbb{R}^m)$ and $\vec{\varphi} \in (C^l(\mathbb{T}^d))^m$, where l is large enough.

The rate $\frac{1}{n^{\frac{d}{2} \wedge 1}}$ is optimal: $\frac{1}{n}$ – lattice discretization error, $\frac{1}{n^{\frac{d}{2}}}$ – particle approximation error

Main tool

Idea of proof: Compare two (time-homogeneous) Markov processes X_t, Y_t taking values in the **same state space** and $X_0 = Y_0 = x$ using

$$\begin{aligned}\mathbb{E}F(X_t) - \mathbb{E}F(Y_t) &= \int_0^t P_s^X (\mathcal{G}^X - \mathcal{G}^Y) P_{t-s}^Y F(x) ds, \\ &= \int_0^t \mathbb{E} [(\mathcal{G}^X - \mathcal{G}^Y) P_{t-s}^Y F(X_s)] ds,\end{aligned}$$

[see e.g. Ethier, Kurtz '86]

Splitting of the problem

Recall

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We consider the Markov processes:

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We will compare:

- $X_t := (\tilde{\rho}_t^n, \tilde{\zeta}_t^n)$ and $Y_t := (\rho_t^{\infty,n}, \zeta_t^{\infty,n})$ [comparison of dynamics]
where the generalized OU process $(\rho_t^{\infty,n}, \zeta_t^{\infty,n})$ started from $(\tilde{\rho}_0^n, \tilde{\zeta}_0^n)$;
- $(\rho_t^{\infty,n}, \zeta_t^{\infty,n})$ and $(\rho_t^\infty, \zeta_t^\infty)$ [comparison of initial conditions]
(both are defined by the same equation).

Generators

We start from the formal computation for cylindrical functions:

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$$\begin{aligned} \mathcal{G}_n^{FF} F(\tilde{\rho}, \tilde{\zeta}) &= \frac{1}{2} \langle \Delta_n \varphi, \tilde{\rho} \rangle + \frac{1}{2} \partial_2 f \langle \Delta_n \varphi, \tilde{\zeta} \rangle \\ &\quad + \frac{1}{4} \partial_2^2 f \langle |\partial_{n,j} \varphi|^2, \tau_j \tilde{\eta} + \tilde{\eta} - 2\widetilde{\eta \tau_j \eta} \rangle + O\left(1/n^{\frac{d}{2}+1}\right), \\ \mathcal{G}^{OU} F(\tilde{\rho}, \tilde{\zeta}) &= \frac{1}{2} \partial_1 f \langle \Delta \varphi, \tilde{\rho} \rangle + \frac{1}{2} \partial_2 f \langle \Delta \varphi, \tilde{\zeta} \rangle \\ &\quad + \frac{1}{2} \partial_2^2 f \langle |\partial_j \varphi|^2, \tilde{\rho} - \tilde{\rho}^2 \rangle \end{aligned}$$

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Thus

$$\begin{aligned} (\mathcal{G}_n^{FF} - \mathcal{G}^{OU}) F(\tilde{\rho}, \tilde{\zeta}) &= \frac{1}{2} \partial_1 f \langle \Delta_n \varphi - \Delta \varphi, \tilde{\rho} \rangle + \frac{1}{2} \partial_2 f \langle \Delta_n \varphi - \Delta \varphi, \tilde{\zeta} \rangle \\ &\quad + \frac{1}{4} \partial_2^2 f \left[\langle |\partial_{n,j} \varphi|^2, \tau_j \tilde{\eta} + \tilde{\eta} - 2\widetilde{\eta \tau_j \eta} \rangle - 2 \langle |\partial_j \varphi|^2, \tilde{\rho} - \tilde{\rho}^2 \rangle \right] + O\left(\frac{1}{n^{\frac{d}{2}+1}}\right) \end{aligned}$$

The difficulties

$$\langle |\partial_{n,j}\varphi|^2, \tau_j\tilde{\eta} + \tilde{\eta} - 2\widetilde{\eta\tau_j\eta} \rangle - 2 \langle |\partial_j\varphi|^2, \tilde{\rho} - \tilde{\rho}^2 \rangle$$

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- $\langle |\partial_{n,j}\varphi|^2, \widetilde{\eta\tau_j\eta} \rangle - \langle |\partial_j\varphi|^2, \tilde{\rho}^2 \rangle$?
- Generators have to be compared on $U := P_{t-s}^{OU}f(\langle\psi, \cdot\rangle, \langle\varphi, \cdot\rangle)$, that **is not cylindrical function**.

The difficulties

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We need different lifting of the particle system to the Sobolev space compatible with its differential structure.

Discrete and continuous Fourier transform

Replace $\tilde{\rho} = \frac{1}{n^d} \sum_{x \in \mathbb{T}_n^d} \rho(x) \delta_x$ and $\tilde{\zeta} = \frac{1}{n^d} \sum_{x \in \mathbb{T}_n^d} \zeta(x) \delta_x$ by a smooth interpolation.

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- Let $L_2(\mathbb{T}_n^d)$ be the Hilbert space of all functions on $\mathbb{T}_n^d$ with inner product

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- $\varsigma_k(x) = e^{2\pi i k \cdot x}$, $k \in \mathbb{Z}^d$, $x \in \mathbb{T}^d \supset \mathbb{T}_n^d$
 - basis vectors on $L_2(\mathbb{T}_n^d)$ and $L_2(\mathbb{T}^d)$, and
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$$L_2(\mathbb{T}_n^d) \ni \rho = \sum_{k \in \mathbb{Z}_n^d} \langle \rho, \varsigma_k \rangle_n \varsigma_k \quad \text{on } \mathbb{T}_n^d, \quad L_2(\mathbb{T}^d) \ni g = \sum_{k \in \mathbb{Z}^d} \langle g, \varsigma_k \rangle \varsigma_k \quad \text{on } \mathbb{T}^d$$

New (smooth) lifting of discrete space

For functions $\rho \in L_2(\mathbb{T}_n^d)$ and $\varphi \in L_2(\mathbb{T}^d)$ define

$$\text{ex}_n \rho := \sum_{k \in \mathbb{Z}_n^d} \langle \rho, \varsigma_k \rangle_n \varsigma_k \quad \text{on } \mathbb{T}^d, \quad \text{pr}_n \varphi := \sum_{k \in \mathbb{Z}_n^d} \langle \varphi, \varsigma_k \rangle \varsigma_k \quad \text{on } \mathbb{T}^d$$

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Basic properties of $\text{ex}_n f$ and $\text{pr}_n g$

- $\text{ex}_n \rho = \rho$ on $\mathbb{T}_n^d$ and $\text{ex}_n \rho \in C^\infty(\mathbb{T}^d)$
- $\text{pr}_n \varphi$ is well defined on $\mathbb{T}_n^d$ for each $\varphi \in H_J$, $J \in \mathbb{R}$.
- $\langle \rho_1, \rho_2 \rangle_n = \langle \text{ex}_n \rho_1, \text{ex}_n \rho_2 \rangle$ and $\langle \rho, \text{pr}_n g \rangle_n = \langle \text{ex}_n \rho, g \rangle$
- $\|\text{pr}_n g - g\|_{H_J} \leq \frac{1}{n} \|g\|_{H_{J+1}}$, $\|\text{ex}_n \varphi - \varphi\|_{H_J} \leq \frac{C}{n} \|\varphi\|_{C^{J+2+\frac{d}{2}}}, \dots$

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- $\|\text{pr}_n g - g\|_{H_J} \leq \frac{1}{n} \|g\|_{H_{J+1}}$, $\|\text{ex}_n \varphi - \varphi\|_{H_J} \leq \frac{C}{n} \|\varphi\|_{C^{J+2+\frac{d}{2}}}, \dots$

$$\begin{aligned} \langle \varphi, \tilde{\rho} \rangle &= \frac{1}{(2n+1)^d} \sum_{x \in \mathbb{T}_n^d} \varphi(x) \rho(x) = \langle \varphi, \rho \rangle_n \\ &= \langle \text{pr}_n \varphi, \rho \rangle_n + O(1/n) = \langle \varphi, \text{ex}_n \rho \rangle + O(1/n) \end{aligned}$$

Comparison of generators for smooth interpolations

We replace

$$\tilde{\rho} \rightsquigarrow \text{ex}_n \rho =: \hat{\rho}, \quad \tilde{\zeta} \rightsquigarrow \text{ex}_n \zeta =: \hat{\zeta}$$

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In generator computations:

$$\langle \Delta_n \text{pr}_n \varphi, \tilde{\zeta} \rangle = \langle \Delta_n \text{pr}_n \varphi, \zeta \rangle_n = \langle \text{ex}_n \Delta_n \text{pr}_n \varphi, \hat{\zeta} \rangle$$

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$$\begin{aligned} (\mathcal{G}_n^{FF} - \mathcal{G}^{OU}) F(\hat{\rho}, \hat{\zeta}) &= \frac{1}{2} \partial_1 f \langle \text{ex}_n \Delta_n \text{pr}_n \psi - \Delta \psi, \hat{\rho} \rangle + \frac{1}{2} \partial_2 f \langle \text{ex}_n \Delta_n \text{pr}_n \varphi - \Delta \varphi, \hat{\zeta} \rangle \\ &+ \frac{1}{4} \partial_2^2 f \left[\langle \text{ex}_n |\partial_{n,j} \text{pr}_n \varphi|^2, \tau_j \eta + \hat{\eta} - 2 \text{ex}_n (\eta \tau_j \eta) \rangle - 2 \langle |\partial_j \varphi|^2, \hat{\rho} - \hat{\rho}^2 \rangle \right] \\ &+ O\left(\frac{1}{n^{\frac{d}{2}+1}}\right). \end{aligned}$$

Overcoming of problems

- $\left| \langle \text{ex}_n \Delta_n \text{pr}_n \varphi - \Delta \varphi, \hat{\zeta} \rangle \right| \leq \frac{1}{n} \|\varphi\|_{H_{l+2}} \|\hat{\zeta}\|_{H_{-l}}$

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- The term $\hat{\rho}^2$ is well defined. Moreover,

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and

$$\text{ex}_n(\eta \tau_j \eta) - \hat{\rho}^2 = \text{ex}_n(\rho \tau_j \rho) - \hat{\rho}^2 + \frac{1}{n^{d/2}} (\text{ex}_n(\rho \tau_j \zeta) + \text{ex}_n(\zeta \tau_j \rho)) + \frac{1}{n^d} \text{ex}_n(\zeta \tau_j \zeta)$$

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- All computations and estimates for $(\mathcal{G}_n^{FF} - \mathcal{G}^{OU}) F(\hat{\rho}, \hat{\zeta})$ can be easily transferred to the case $F \in C^{1,3}(H_J \times H_{-l}).$

Differentiability of $P_t^{OU} F(\hat{\rho}, \hat{\zeta})$

A solution to

$$d\rho_t^\infty = \frac{1}{2} \Delta \rho_t^\infty dt$$

$$d\zeta_t^\infty = \frac{1}{2} \Delta \zeta_t^\infty dt + \nabla \cdot \left(\sqrt{\rho_t^\infty (1 - \rho_t^\infty)} dW_t \right)$$

exists for all $\rho_0^\infty \in L_2(\mathbb{T}^d; [0, 1])$ and $\zeta_0^\infty \in H_{-l}$ for $l > \frac{d}{2} + 1$.

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For $F \in C(H_J \times H_{-l})$ (e.g. $F = f(\langle \psi, \cdot \rangle, \langle \varphi, \cdot \rangle)$) define $U_t(\rho_0^\infty, \zeta_0^\infty) := \mathbb{E} F(\rho_t^\infty, \zeta_t^\infty)$

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Proposition [Gess, K. '24]

Let $l > \frac{d}{2} + 1$ and $F \in C_b^{2,4}(H_{-l})$. Then $U_t(\rho_0^\infty, \zeta_0^\infty) = \mathbb{E} F(\zeta_t^\infty) \in C_b^{1,3}(H_J \times H_{-l})$ for $J > \frac{d}{2}$. Moreover,

$$D_1 U_t(\rho_0^\infty, \zeta_0^\infty)[h] = \frac{1}{2} \mathbb{E} [D^2 F(\zeta_t^\infty) : DV_t(\rho_0^\infty)[h]]$$

with

$$\begin{aligned} V_t(\rho_0^\infty)(\varphi, \psi) &= \text{Cov}(\langle \varphi, \zeta_t^\infty \rangle, \langle \psi, \zeta_t^\infty \rangle) \\ &= \frac{1}{2} \int_0^t \langle \nabla P_{t-s}^{HE} \varphi \cdot \nabla P_{t-s}^{HE} \psi, \rho_s^\infty (1 - \rho_s^\infty) \rangle ds \end{aligned}$$

Berry-Esseen bound for the initial fluctuations

- It remains only to compare

$$\mathbb{E}F(\rho_t^{\infty,n}, \zeta_t^{\infty,n}) - \mathbb{E}F(\rho_t^{\infty}, \zeta_t^{\infty}) = P_t^{OU} F(\hat{\rho}_0^n, \hat{\zeta}_0^n) - P_t^{OU} F(\rho_0, \zeta_0)$$

where ρ_t^{∞} started from the initial profile ρ_0 and ζ_t^{∞} started from the centered Gaussian distribution with

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- It is enough to compare only

$$\mathbb{E}G(\text{ex}_n \zeta_0^n) - \mathbb{E}G(\text{pr}_n \zeta_0),$$

where $G \in C^3(H_{-I})$, where

$$\text{ex}_n \zeta_0^n := \sum_{k \in \mathbb{Z}_n^d} \langle \zeta_0^n, s_k \rangle_n s_k, \quad \text{pr}_n \zeta_0 := \sum_{k \in \mathbb{Z}_n^d} \langle \zeta_0, s_k \rangle s_k$$

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- Is enough to compare for $g \in C^3(\mathbb{R}^{\mathbb{Z}_n^d})$

$$\mathbb{E}g\left(\left((1 + |k|^2)^{-1/2} \langle \zeta_0^n, s_k \rangle_n\right)_{k \in \mathbb{Z}_n^d}\right) - \mathbb{E}g\left(\left((1 + |k|^2)^{-1/2} \langle \zeta_0, s_k \rangle\right)_{k \in \mathbb{Z}_n^d}\right).$$

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$$\text{ex}_n \zeta_0^n := \sum_{k \in \mathbb{Z}_n^d} \langle \zeta_0^n, s_k \rangle_n s_k, \quad \text{pr}_n \zeta_0 := \sum_{k \in \mathbb{Z}_n^d} \langle \zeta_0, s_k \rangle s_k$$

- Is enough to compare for $g \in C^3(\mathbb{R}^{\mathbb{Z}_n^d})$

$$\mathbb{E}g\left(\left((1 + |k|^2)^{-1/2} \langle \zeta_0^n, s_k \rangle_n\right)_{k \in \mathbb{Z}_n^d}\right) - \mathbb{E}g\left(\left((1 + |k|^2)^{-1/2} \langle \zeta_0, s_k \rangle\right)_{k \in \mathbb{Z}_n^d}\right).$$

- Apply multidimensional Berry-Esseen theorem [e.g., Meckes '09]

References

- [1] Benjamin Gess and Vitalii Konarovskiy. A quantitative central limit theorem for the simple symmetric exclusion process (2024), arXiv:2408.01238

Thank you!