A Quantitative Central Limit Theorem for Simple Symmetric Exclusion Process

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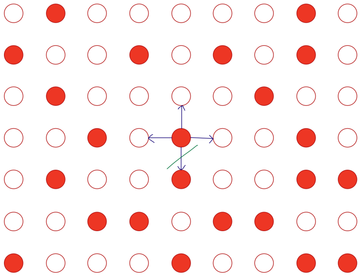
Simple symmetric exclusion process

On the d-dim discrete torus

$$\mathbb{T}_n^d := \left\{ \frac{k}{n} : \ k \in \mathbb{Z}_n^d := \{0, \ldots, n-1\}^d \right\} \subset \mathbb{T}^d = (\mathbb{R}/\mathbb{Z})^d$$

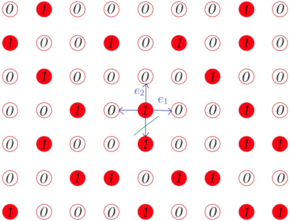
we consider a **Simple Symmetric Exclusion Process** (SSEP)



State space and generator

Particle configuration $\eta \in \{0,1\}^{\mathbb{T}_n^d}$:

$$\eta(x) = 0 \Leftrightarrow \text{side } x \text{ is empty}$$
$$\eta(x) = 1 \Leftrightarrow \text{side } x \text{ is occupied}$$



$$\eta^{x\leftrightarrow y}(z) = \begin{cases} \eta(z), & z \neq x, y, \\ \eta(y), & z = x, \\ \eta(x), & z = y, \end{cases}$$

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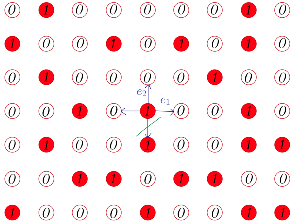

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$$\mathcal{G}^n F(\eta) := \frac{n^2}{2} \sum_{j=1}^{d} \sum_{x \in \mathbb{T}_n} \left[F(\eta^{x\leftrightarrow x+e_j}) - F(\eta) \right]$$ [Kipnis, Landim '99]

SSEP is already parabolically rescaled: space $\sim \frac{1}{n}$ time $\sim n^2$!

Non-equilibrium SSEP

Let η_t^n, $t \geq 0$, be a SSEP and $\rho_0 : \mathbb{T}^d \to [0,1]$ be an initial profile.

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Let η^n_t, $t \geq 0$, be a SSEP and $\rho_0 : \mathbb{T}^d \to [0,1]$ be an initial profile.

Assume that $\eta^n_0(x) \sim \mathrm{Bernulli}(\rho_0(x))$, $x \in \mathbb{T}^d_n$, are independent.

The process $\rho^n_t(x) := \mathbb{E}\eta^n_t(x)$ solves the discrete stochastic Heat equation

$$d\rho^n_t(x) = \frac{1}{2}\Delta_n \rho^n_t(x) dt$$

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Thus,

$$\rho_t^n := \frac{1}{n^d} \sum_{x \in \mathbb{T}_n^d} \rho_t^n(x)\delta_x \to \rho_t,$$

where ρ_t solves

$$d\rho_t = \frac{1}{2}\Delta \rho_t dt$$

with initial condition ρ_0.

Law of large numbers

Theorem [see e.g. in Kipnis, Landim '99]

Let $\rho_0 : \mathbb{T}^d \to [0,1]$ be an initial density profile and $\eta_0^n(x) \sim \mathrm{Bernulli}(\rho_0(x))$ be independent. Then

$$\eta_t^n := \frac{1}{n^d} \sum_{x \in \mathbb{T}_n^d} \eta_t^n(x) \delta_x, \quad t \geq 0$$

converges in probability to $\rho_t(x)$, $t \geq 0$, where ρ_t solves

$$d\rho_t = \frac{1}{2} \Delta \rho_t dt$$

with initial condition ρ_0.

Convergence of generator

Note that $\langle \varphi, \eta_t^n \rangle$ solves the martingale problem

$$f\left(\langle \varphi, \eta_t^n \rangle\right) - \int_0^t \mathcal{G}_n^{EP} f\left(\langle \varphi, \eta_s^n \rangle\right) ds \quad \text{is a mart.,}$$

where

$$\mathcal{G}^n f\left(\langle \varphi, \eta \rangle\right) := \frac{n^2}{2} \sum_{j=1}^{d} \sum_{x \in \mathbb{T}_n} f'\left(\langle \varphi, \eta \rangle\right) \underbrace{\left(\left\langle \varphi, \eta^{x \leftrightarrow x+e_j} \right\rangle - \langle \varphi, \eta \rangle\right)}_{\left[\varphi(x+e_j)-\varphi(x)\right]\left[\eta(x)-\eta(x+e_j)\right]}$$

$$+ \frac{n^2}{4} \sum_{j=1}^{d} \sum_{x \in \mathbb{T}_n} f''\left(\langle \varphi, \eta \rangle\right) \left(\left\langle \varphi, \eta^{x \leftrightarrow x+e_j} \right\rangle - \langle \varphi, \eta \rangle\right)^2$$

$$= \frac{1}{2} f'\left(\langle \varphi, \eta \rangle\right) \langle \Delta_n \varphi, \eta \rangle$$

$$+ \frac{1}{4n^d} f''\left(\langle \varphi, \eta \rangle\right) \sum_{j=1}^{d} \left\langle \left|\partial_{n,j} \varphi\right|^2, \tau_j \eta + \eta - 2\eta \tau_j \eta \right\rangle + O\left(\frac{\|f'''\|}{n^{2d+1}}\right)$$

Density fluctuation field and CLT

We now consider the fluctuations of the SSEP around its mean:

$$\zeta_t^n := n^{\frac{d}{2}} \left(\eta_t^n - \rho_t^n\right).$$

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The generator of ζ_t^n can be expanded as follows

$$\mathcal{G}^n f \left(\langle \varphi, \zeta \rangle \right) = \frac{1}{2} f' \left(\langle \varphi, \zeta \rangle \right) \langle \Delta_n \varphi, \tilde{\zeta} \rangle + \frac{n^d}{4n^d} f'' \left(\langle \varphi, \zeta \rangle \right) \left\langle \left| \nabla_n \varphi \right|^2, \tau\eta + \eta - 2\eta\tau\eta \right\rangle + O\left(1/n^{\frac{d}{2}+1} \right)$$

$$\xrightarrow[n\to\infty]{}$$

$$\mathcal{G} f \left(\langle \varphi, \zeta \rangle \right) = \frac{1}{2} f' \left(\langle \varphi, \zeta \rangle \right) \langle \Delta \varphi, \zeta \rangle + \frac{1}{4} f'' \left(\langle \varphi, \zeta \rangle \right) \left\langle \left| \nabla \varphi \right|^2, \rho + \rho - 2\rho^2 \right\rangle$$

Central limit theorem

Theorem 2 **[Galves, Kipnis, Spohn; Ravishankar '90]**

Let the initial density profile ρ_0 be smooth. Then the density fluctuation field

$$\zeta_t^n := \frac{1}{n^d} \sum_{x \in \mathbb{T}_n^d} \zeta_t(x) \delta_x$$

converges in $D([0, T], \mathcal{D}')$ to the generalized Ornstein-Uhlenbeck process that solves the linear SPDE

$$d\zeta_t = \frac{1}{2} \Delta \zeta_t dt + \nabla \cdot \left(\sqrt{\rho_t(1-\rho_t)} dW_t \right)$$

with the centered Gaussian initial condition such that

$$\mathbb{E}\left[\langle \zeta_0, \varphi \rangle^2 \right] = \langle \rho_0(1-\rho_0)\varphi, \varphi \rangle$$

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- [Gess, Wu, Zhang '24; Djurdjevac, Gerencsér, Kremp '24, Clini, Fehrman '25]: Higher order approximation of SPDEs.
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- [Chassagneux, Szpruch, Tse '22]: Weak quantitative propagation of chaos (mean field limit)
- [Kolokoltsov '10] Central limit theorem for the Smoluchovski coagulation model (mean field limit, non-local Smoluchowski's coagulation equation)
- ...

Main result

Theorem 3 [Gess, K. '24]

Let

- the initial density profile $\rho_0 : \mathbb{T}^d \to [0,1]$ be smooth enough,
- η^n_t be SSEP with $\eta^n_0(x) \sim \mathrm{Bernulli}(\rho_0(x))$ and independent.

Then

$$\sup_{t\in[0,T]} \left|\mathbb{E}f\left(\langle\vec{\varphi},\zeta^n_t\rangle\right) - \mathbb{E}f\left(\langle\vec{\varphi},\zeta_t\rangle\right)\right| \leq \frac{C}{n^{\frac{d}{2}\wedge 1}} \|f\|_{\mathrm{C}^3_l} \|\vec{\varphi}\|_{\mathrm{C}^l}$$

for all $n \geq 1$, $f \in \mathrm{C}^3_b(\mathbb{R}^m)$ and $\vec{\varphi} \in \left(\mathrm{C}^l(\mathbb{T}^d)\right)^m$, where l is large enough.

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The rate $\frac{1}{n^{\frac{d}{2}\wedge 1}}$ is optimal:

$\frac{1}{n}$ – lattice discretization error, $\frac{1}{n^{\frac{d}{2}}}$ – particle approximation error

Main tool

Idea of proof: Compare two (time-homogeneous) Markov processes X_t, Y_t using

$$\mathbb{E}F(X_t) - \mathbb{E}F(Y_t) = \mathbb{E}U_t(X_0) - \mathbb{E}U_t(Y_0) + \int_0^t \mathbb{E}\left[\left(\mathcal{G}^X - \mathcal{G}^Y\right) U_{t-s}(X_s)\right] ds,$$

where $U_t(x) = \mathbb{E}_x F(Y_t)$.

[see e.g. Ethier, Kurtz '86]

Comparison of semigroup

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- solution to heat equation and generalized OU process $Y = (\rho_t, \zeta_t)$.

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where $U_t(\rho, \zeta) = \mathbb{E}_{\rho,\zeta} F(\rho_t, \zeta_t)$

Generators

We start from the formal computation for cylindrical functions:

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Thus

$$(\mathcal{G}^n-\mathcal{G})\,U(\rho,\zeta) = \dots + \frac{1}{4}\partial_2^2 f\left[\left\langle|\nabla_n\varphi|^2, \tau\eta+\eta-2\eta\tau\eta\right\rangle - \left\langle|\nabla\varphi|^2, 2\rho-2\rho^2\right\rangle\right] + O\left(\frac{1}{n^{\frac{d}{2}+1}}\right).$$

The difficulties

$$\left\langle |\nabla_n \varphi|^2, \tau\eta + \eta - 2\eta\tau\eta \right\rangle - \left\langle |\nabla\varphi|^2, 2\rho - 2\rho^2 \right\rangle$$

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We need different lifting of the particle system to the Sobolev spaces.

Discrete and continuous Fourier transform

We will replace $\rho = \frac{1}{n^d} \sum_{x \in \mathbb{T}_n^d} \rho(x)\delta_x$ and $\zeta = \frac{1}{n^d} \sum_{x \in \mathbb{T}_n^d} \zeta(x)\delta_x$ by a smooth interpolation.

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- Let $L_2(\mathbb{T}_n^d)$ be the Hilbert space of all functions on $\mathbb{T}_n^d$ with inner product

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- $\varsigma_k(x) = e^{2\pi i k \cdot x}$, $k \in \mathbb{Z}^d$, $x \in \mathbb{T}^d \supset \mathbb{T}_n^d$
  – basis vectors on $L_2(\mathbb{T}_n^d)$ and $L_2(\mathbb{T}^d)$, and
  – eigenvectors for discrete and continuous diff. operators

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$$L_2(\mathbb{T}_n^d) \ni \rho = \sum_{k \in \mathbb{Z}_n^d} \langle \rho, \varsigma_k \rangle_n \varsigma_k \ \text{ on } \mathbb{T}_n^d, \quad L_2(\mathbb{T}^d) \ni g = \sum_{k \in \mathbb{Z}^d} \langle g, \varsigma_k \rangle_{L_2} \varsigma_k \ \text{ on } \mathbb{T}^d$$

New (smooth) lifting of discrete space

For functions $\rho \in L_2(\mathbb{T}_n^d)$ and $\varphi \in L_2(\mathbb{T}^d)$ define

$$\mathrm{ex}_n \rho := \sum_{k \in \mathbb{Z}_n^d} \langle \rho, \varsigma_k \rangle_n \varsigma_k \quad \text{on } \mathbb{T}^d, \quad \mathrm{pr}_n \varphi := \sum_{k \in \mathbb{Z}_n^d} \langle \varphi, \varsigma_k \rangle_{L_2} \varsigma_k \quad \text{on } \mathbb{T}^d$$

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Basic properties of $\mathrm{ex}_n f$ and $\mathrm{pr}_n g$

- $\mathrm{ex}_n\rho = \rho$ on $\mathbb{T}_n^d$ and $\mathrm{ex}_n\rho \in \mathrm{C}^\infty(\mathbb{T}^d)$
- $\mathrm{pr}_n\varphi$ is well defined on $\mathbb{T}_n^d$ for each $\varphi \in H_J$, $J \in \mathbb{R}$.
- $\langle \rho_1, \rho_2\rangle_n = \langle \mathrm{ex}_n\rho_1, \mathrm{ex}_n\rho_2\rangle_{L_2}$ and $\langle \rho, \mathrm{pr}_n g\rangle_n = \langle \mathrm{ex}_n\rho, g\rangle_{L_2}$
- $\|\mathrm{pr}_n g - g\|_{H_J} \leq \frac{1}{n}\|g\|_{H_{J+1}}$, $\|\mathrm{ex}_n\varphi - \varphi\|_{H_J} \leq \frac{C}{n}\|\varphi\|_{\mathrm{C}^{J+2+\frac{d}{2}}}, \dots$

Comparison of generators for smooth interpolations

We replace

$$\frac{1}{n^d}\sum_{x\in\mathbb{T}_n^d}\rho(x)\delta_x \rightsquigarrow \mathrm{ex}_n\rho =: \hat{\rho}, \qquad \frac{1}{n^d}\sum_{x\in\mathbb{T}_n^d}\zeta(x)\delta_x \rightsquigarrow \mathrm{ex}_n\zeta =: \hat{\zeta}$$

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Thus

$$\begin{aligned}(\mathcal{G}^n - \mathcal{G})F(\mathrm{ex}_n\rho, \mathrm{ex}_n\rho) &= \frac{1}{2}\partial_1 f[\ldots] \\ &+ \frac{1}{4}\partial_2^2 f\left[\left\langle \mathrm{ex}_n\left|\nabla_n \mathrm{pr}_n\varphi\right|^2, \hat{\tau\eta} + \hat{\eta} - 2\mathrm{ex}_n(\eta\tau\eta)\right\rangle - \left\langle \left|\nabla\varphi\right|^2, 2\hat{\rho} - 2\hat{\rho}^2\right\rangle\right] \\ &+ O\left(\frac{1}{n^{\frac{d}{2}+1}}\right).\end{aligned}$$

Overcoming of problems

We deal with the most problematic term $\mathrm{ex}_n(\eta\tau\eta) - \hat{\rho}^2$ as follows

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- The term $\mathbb{E}\left\langle \frac{1}{n^d}\,\mathrm{ex}_n\left(\eta_t^n - \rho_t^n\right)\tau\left(\eta_t^n - \rho_t^n\right), G(\eta_t^n)\right\rangle^2$ can be controlled via 4-point correlation function

$$\mathbb{E}\left[\prod_{i=1}^{4}(\eta_t^n(x_i) - \rho_t^n(x_i))\right] \lesssim \frac{1}{n^2}$$

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All computations and estimates for $(\mathcal{G}^n - \mathcal{G})\,F(\hat{\rho}, \hat{\zeta})$ can be easily transferred to the case $F \in \mathrm{C}^{1,3}(H_J \times H_{-I})$.

Differentiability of $U_t(\rho, \zeta) = \mathbb{E}_{\rho,\zeta} F(\rho_t, \zeta_t)$

A solution to

$$d\rho_t = \frac{1}{2}\Delta\rho_t dt$$
$$d\zeta_t = \frac{1}{2}\Delta\zeta_t dt + \nabla \cdot \left(\sqrt{\rho_t(1-\rho_t)} dW_t\right)$$

exists for all $\rho_0 \in L_2(\mathbb{T}^d; [0,1])$ and $\zeta_0 \in H_{-l}$ for $l > \frac{d}{2} + 1$.

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Proposition [Gess, K. '24]

Let $l > \frac{d}{2}+1$ and $F \in \mathrm{C}_b^{2,4}(H_{-l})$. Then $U_t(\rho,\zeta) = \mathbb{E}F(\zeta_t) \in \mathrm{C}_b^{1,3}(H_J \times H_{-l})$ for $J > \frac{d}{2}$. Moreover,

$$D_1 U_t(\rho,\zeta) = \frac{1}{2}\mathbb{E}\left[D^2 F(\zeta_t) : DV_t(\rho)\right]$$

with

$$V_t(\rho)(\varphi,\varphi) = \mathrm{Var}\,\langle\varphi,\zeta_t\rangle = \frac{1}{2}\int_0^t \left\langle |\nabla\varphi_{t-s}|^2, \rho_s(1-\rho_s)\right\rangle ds$$

and φ_t is a solution to the heat equation started from φ.

Berry-Esseen bound for the initial fluctuations

- It remains only to compare

$$\mathbb{E}U_t(\mathrm{ex}_n\rho_0^n, \mathrm{ex}_n\zeta_0^n) - \mathbb{E}U_t(\rho_0, \zeta_0)$$

where $\zeta_0^n = n^{d/2}(\eta_0^n - \rho_0^n)$, $\rho_0^n = \rho_0\big|_{\mathbb{T}_n^d}$, $\eta_0^n(x) \sim \mathrm{Bernulli}(\rho_0(x))$, $x \in \mathbb{T}_n^d$, i.i.d. and ζ_0 is a centered Gaussian distribution with

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where $G \in \mathrm{C}^3(H_{-l})$, where

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$$\mathbb{E} g\left(\left((1+|k|^2)^{-l/2} \langle \zeta_0^n, \varsigma_k \rangle_n \right)_{k \in \mathbb{Z}_n^d} \right) - \mathbb{E} g\left(\left((1+|k|^2)^{-l/2} \langle \zeta_0, \varsigma_k \rangle \right)_{k \in \mathbb{Z}_n^d} \right).$$

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- Apply multidimensional Berry-Essen theorem [e.g., Meckes '09]

References

[1] Benjamin Gess and Vitalii Konarovskyi. A quantitative central limit theorem for the simple symmetric exclusion process (2024), arXiv:2408.01238

Thank you!