

A Quantitative Central Limit Theorem for Simple Symmetric Exclusion Process

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From discrete systems to SPDEs – Vienna

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Universität Hamburg

DER FORSCHUNG | DER LEHRE | DER BILDUNG

LLN and CLT

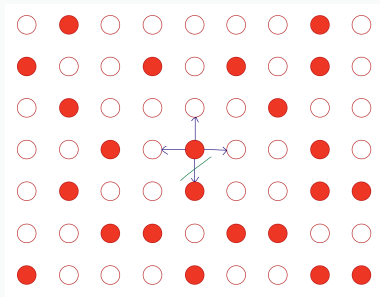
Quantitative CLT for SSEP

Simple symmetric exclusion process

On the d -dim discrete torus

$$\mathbb{T}_n^d := \left\{ \frac{k}{n} : k \in \mathbb{Z}_n^d := \{0, \dots, n-1\}^d \right\} \subset \mathbb{T}^d = (\mathbb{R}/\mathbb{Z})^d$$

we consider a **Simple Symmetric Exclusion Process** (SSEP)

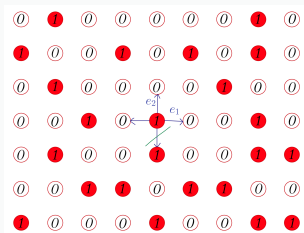


State space and generator

Particle configuration $\eta \in \{0, 1\}^{\mathbb{T}_n^d}$:

$\eta(x) = 0 \Leftrightarrow$ side x is empty

$\eta(x) = 1 \Leftrightarrow$ side x is occupied



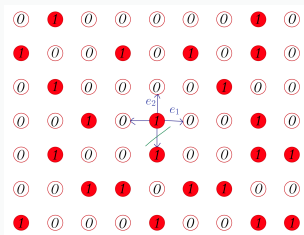
$$\eta^{x \leftrightarrow y}(z) = \begin{cases} \eta(z), & z \neq x, y, \\ \eta(y), & z = x, \\ \eta(x), & z = y, \end{cases}$$

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$$\mathcal{G}^n F(\eta) := \frac{n^2}{2} \sum_{j=1}^d \sum_{x \in \mathbb{T}_n} [F(\eta^{x \leftrightarrow x+e_j}) - F(\eta)] \quad [\text{Kipnis, Landim '99}]$$

SSEP is already parabolically rescaled: space $\sim \frac{1}{n}$ time $\sim n^2$!

Non-equilibrium SSEP

Let $\eta_t^n, t \geq 0$, be a SSEP and $\rho_0 : \mathbb{T}^d \rightarrow [0, 1]$ be an initial profile.

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The process $\rho_t^n(x) := \mathbb{E}\eta_t^n(x)$ solves the discrete stochastic Heat equation

$$d\rho_t^n(x) = \frac{1}{2}\Delta_n \rho_t^n(x)dt$$

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Thus,

$$\rho_t^n := \frac{1}{n^d} \sum_{x \in \mathbb{T}_n^d} \rho_t^n(x) \delta_x \rightarrow \rho_t,$$

where ρ_t solves

$$d\rho_t = \frac{1}{2}\Delta \rho_t dt$$

with initial condition ρ_0 .

Law of large numbers

Theorem [see e.g. in Kipnis, Landim '99]

Let $\rho_0 : \mathbb{T}^d \rightarrow [0, 1]$ be an initial density profile and $\eta_0^n(x) \sim B(\rho_0(x))$ be independent. Then

$$\eta_t^n := \frac{1}{n^d} \sum_{x \in \mathbb{T}_n^d} \eta_t^n(x) \delta_x, \quad t \geq 0$$

converges in probability to $\rho_t(x)$, $t \geq 0$, where ρ_t solves

$$d\rho_t = \frac{1}{2} \Delta \rho_t dt$$

with initial condition ρ_0 .

Convergence of generator

Note that $\langle \varphi, \eta_t^n \rangle$ solves the martingale problem

$$f(\langle \varphi, \eta_t^n \rangle) - \int_0^t \mathcal{G}_n^{EP} f(\langle \varphi, \eta_s^n \rangle) ds \quad \text{is a mart.},$$

where

$$\begin{aligned} \mathcal{G}_n f(\langle \varphi, \eta \rangle) &:= \frac{n^2}{2} \sum_{j=1}^d \sum_{x \in \mathbb{T}_n} f'(\langle \varphi, \eta \rangle) \underbrace{(\langle \varphi, \eta^{x \leftrightarrow x+e_j} \rangle - \langle \varphi, \eta \rangle)}_{\frac{1}{n^d} [\varphi(x+e_j) - \varphi(x)] [\eta(x) - \eta(x+e_j)]} \\ &\quad + \frac{n^2}{4} \sum_{j=1}^d \sum_{x \in \mathbb{T}_n} f''(\langle \varphi, \eta \rangle) (\langle \varphi, \eta^{x \leftrightarrow x+e_j} \rangle - \langle \varphi, \eta \rangle)^2 \\ &= \frac{1}{2} f'(\langle \varphi, \eta \rangle) \langle \Delta_n \varphi, \eta \rangle \\ &\quad + \frac{1}{4n^d} f''(\langle \varphi, \eta \rangle) \sum_{j=1}^d \langle |\partial_{n,j} \varphi|^2, \tau_j \eta + \eta - 2\eta \tau_j \eta \rangle + O\left(\frac{\|f'''\|}{n^{2d+1}}\right) \end{aligned}$$

Density fluctuation field and CLT

We now consider the fluctuations of the SSEP around its mean:

$$\zeta_t^n := n^{\frac{d}{2}} (\eta_t^n - \rho_t^n).$$

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The generator of ζ_t^n can be expanded as follows

$$\begin{aligned} \mathcal{G}^n f(\langle \varphi, \zeta \rangle) &= \frac{1}{2} f'(\langle \varphi, \zeta \rangle) \langle \Delta_n \varphi, \tilde{\zeta} \rangle + \frac{n^d}{4n^d} f''(\langle \varphi, \zeta \rangle) \langle |\nabla_n \varphi|^2, \tau \eta + \eta - 2\eta \tau \eta \rangle \\ &\quad + O\left(1/n^{\frac{d}{2}+1}\right) \end{aligned}$$

$$\xrightarrow[n \rightarrow \infty]{} \quad$$

$$\mathcal{G} f(\langle \varphi, \zeta \rangle) = \frac{1}{2} f'(\langle \varphi, \zeta \rangle) \langle \Delta \varphi, \zeta \rangle + \frac{1}{4} f''(\langle \varphi, \zeta \rangle) \langle |\nabla \varphi|^2, \rho + \rho - 2\rho^2 \rangle$$

Central limit theorem

Theorem 2 [Galves, Kipnis, Spohn; Ravishankar '90]

Let the initial density profile ρ_0 be smooth. Then the density fluctuation field

$$\zeta_t^n := \frac{1}{n^d} \sum_{x \in \mathbb{T}_n^d} \zeta_t^n(x) \delta_x$$

converges in $D([0, T], \mathcal{D}')$ to the generalized Ornstein-Uhlenbeck process that solves the linear SPDE

$$d\zeta_t = \frac{1}{2} \Delta \zeta_t dt + \nabla \cdot \left(\sqrt{\rho_t(1 - \rho_t)} dW_t \right)$$

with the centered Gaussian initial condition such that

$$\mathbb{E} [\langle \zeta_0, \varphi \rangle^2] = \langle \rho_0(1 - \rho_0) \varphi, \varphi \rangle$$

Our goal

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- [Chassagneux, Szpruch, Tse '22]: Weak quantitative propagation of chaos
(mean field limit)
- [Kolokoltsov '10] Central limit theorem for the Smoluchovski coagulation model
(mean field limit, non-local Smoluchowski's coagulation equation)
- ...

Main result

Theorem 3 [Gess, K. '24]

Let

- the initial density profile $\rho_0 : \mathbb{T}^d \rightarrow [0, 1]$ be smooth enough,
- η_t^n be SSEP with $\eta_0^n(x) \sim B(\rho_0(x))$ and independent.

Then

$$\sup_{t \in [0, T]} |\mathbb{E} f(\langle \vec{\varphi}, \zeta_t^n \rangle) - \mathbb{E} f(\langle \vec{\varphi}, \zeta_t \rangle)| \leq \frac{C}{n^{\frac{d}{2} \wedge 1}} \|f\|_{C_t^3} \|\vec{\varphi}\|_{C^I}$$

for all $n \geq 1$, $f \in C_b^3(\mathbb{R}^m)$ and $\vec{\varphi} \in (C^I(\mathbb{T}^d))^m$, where I is large enough.

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for all $n \geq 1$, $f \in C_b^3(\mathbb{R}^m)$ and $\vec{\varphi} \in (C^I(\mathbb{T}^d))^m$, where I is large enough.

The rate $\frac{1}{n^{\frac{d}{2} \wedge 1}}$ is optimal:

$\frac{1}{n}$ – lattice discretization error, $\frac{1}{n^{\frac{d}{2}}}$ – particle approximation error

Main tool

Idea of proof: Compare two (time-homogeneous) Markov processes X_t, Y_t using

$$\begin{aligned}\mathbb{E}F(X_t) - \mathbb{E}F(Y_t) &= \mathbb{E}U_t(X_0) - \mathbb{E}U_t(Y_0) \\ &\quad + \int_0^t \mathbb{E} \left[(\mathcal{G}^X - \mathcal{G}^Y) U_{t-s}(X_s) \right] ds,\end{aligned}$$

where $U_t(x) = \mathbb{E}_x F(Y_t)$.

[see e.g. Ethier, Kurtz '86]

Comparison of semigroup

We consider the Markov processes:

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- solution to heat equation and generalized OU process $Y = (\rho_t, \zeta_t)$.

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where $U_t(\rho, \zeta) = \mathbb{E}_{\rho, \zeta} F(\rho_t, \zeta_t)$

Comparison of dynamics

Quantitative CLT for SSEP

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We start from the formal computation for cylindrical functions:

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$$\mathcal{G}^n U(\rho, \zeta) = \dots + \frac{1}{4} \partial_2^2 f \langle |\nabla_n \varphi|^2, \tau\eta + \eta - 2\eta\tau\eta \rangle + O\left(1/n^{\frac{d}{2}+1}\right),$$

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Thus

$$\begin{aligned} (\mathcal{G}^n - \mathcal{G}) U(\rho, \zeta) &= \dots + \frac{1}{4} \partial_2^2 f \left[\langle |\nabla_n \varphi|^2, \tau\eta + \eta - 2\eta\tau\eta \rangle - \langle |\nabla \varphi|^2, 2\rho - 2\rho^2 \rangle \right] \\ &\quad + O\left(\frac{1}{n^{\frac{d}{2}+1}}\right). \end{aligned}$$

The difficulties

$$\langle |\nabla_n \varphi|^2, \tau\eta + \eta - 2\eta\tau\eta \rangle - \langle |\nabla \varphi|^2, 2\rho - 2\rho^2 \rangle$$

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We need different lifting of the particle system to the Sobolev spaces.

Discrete and continuous Fourier transform

We will replace $\rho = \frac{1}{n^d} \sum_{x \in \mathbb{T}_n^d} \rho(x) \delta_x$ and $\zeta = \frac{1}{n^d} \sum_{x \in \mathbb{T}_n^d} \zeta(x) \delta_x$ by a smooth interpolation.

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- Let $L_2(\mathbb{T}_n^d)$ be the Hilbert space of all functions on $\mathbb{T}_n^d$ with inner product

$$\langle \rho_1, \rho_2 \rangle_n := \frac{1}{n^d} \sum_{x \in \mathbb{T}_n^d} \rho_1(x) \rho_2(x)$$

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- $\varsigma_k(x) = e^{2\pi i k \cdot x}$, $k \in \mathbb{Z}^d$, $x \in \mathbb{T}^d \supset \mathbb{T}_n^d$
 - basis vectors on $L_2(\mathbb{T}_n^d)$ and $L_2(\mathbb{T}^d)$, and
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$$L_2(\mathbb{T}_n^d) \ni \rho = \sum_{k \in \mathbb{Z}_n^d} \langle \rho, \varsigma_k \rangle_n \varsigma_k \text{ on } \mathbb{T}_n^d, \quad L_2(\mathbb{T}^d) \ni g = \sum_{k \in \mathbb{Z}^d} \langle g, \varsigma_k \rangle_{L_2} \varsigma_k \text{ on } \mathbb{T}^d$$

New (smooth) lifting of discrete space

For functions $\rho \in L_2(\mathbb{T}_n^d)$ and $\varphi \in L_2(\mathbb{T}^d)$ define

$$\text{ex}_n \rho := \sum_{k \in \mathbb{Z}_n^d} \langle \rho, \varsigma_k \rangle_n \varsigma_k \quad \text{on } \mathbb{T}^d, \quad \text{pr}_n \varphi := \sum_{k \in \mathbb{Z}_n^d} \langle \varphi, \varsigma_k \rangle_{L_2} \varsigma_k \quad \text{on } \mathbb{T}^d$$

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Basic properties of $\text{ex}_n f$ and $\text{pr}_n g$

- $\text{ex}_n \rho = \rho$ on $\mathbb{T}_n^d$ and $\text{ex}_n \rho \in C^\infty(\mathbb{T}^d)$
- $\text{pr}_n \varphi$ is well defined on $\mathbb{T}_n^d$ for each $\varphi \in H_J$, $J \in \mathbb{R}$.
- $\langle \rho_1, \rho_2 \rangle_n = \langle \text{ex}_n \rho_1, \text{ex}_n \rho_2 \rangle_{L_2}$ and $\langle \rho, \text{pr}_n g \rangle_n = \langle \text{ex}_n \rho, g \rangle_{L_2}$
- $\|\text{pr}_n g - g\|_{H_J} \leq \frac{1}{n} \|g\|_{H_{J+1}}$, $\|\text{ex}_n \varphi - \varphi\|_{H_J} \leq \frac{C}{n} \|\varphi\|_{C^{J+2+\frac{d}{2}}}, \dots$

Comparison of generators for smooth interp.

We replace

$$\frac{1}{n^d} \sum_{x \in \mathbb{T}_n^d} \rho(x) \delta_x \rightsquigarrow \text{ex}_n \rho =: \hat{\rho}, \quad \frac{1}{n^d} \sum_{x \in \mathbb{T}_n^d} \zeta(x) \delta_x \rightsquigarrow \text{ex}_n \zeta =: \hat{\zeta}$$

in particular

$$\langle \varphi, \text{ex}_n \zeta \rangle_{L_2} = \langle \text{pr}_n \varphi, \zeta \rangle_n = \langle \text{pr}_n \varphi, \zeta \rangle$$

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$$\frac{1}{n^d} \sum_{x \in \mathbb{T}_n^d} \rho(x) \delta_x \rightsquigarrow \text{ex}_n \rho =: \hat{\rho}, \quad \frac{1}{n^d} \sum_{x \in \mathbb{T}_n^d} \zeta(x) \delta_x \rightsquigarrow \text{ex}_n \zeta =: \hat{\zeta}$$

in particular

$$\langle \varphi, \text{ex}_n \zeta \rangle_{L_2} = \langle \text{pr}_n \varphi, \zeta \rangle_n = \langle \text{pr}_n \varphi, \zeta \rangle$$

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Thus

$$\begin{aligned} (\mathcal{G}^n - \mathcal{G})F(\text{ex}_n \rho, \text{ex}_n \rho) &= \frac{1}{2} \partial_1 f[\dots] \\ &+ \frac{1}{4} \partial_2^2 f \left[\langle \text{ex}_n |\nabla_n \text{pr}_n \varphi|^2, \hat{\tau} \eta + \hat{\eta} - 2 \text{ex}_n(\eta \tau \eta) \rangle - \langle |\nabla \varphi|^2, 2\hat{\rho} - 2\hat{\rho}^2 \rangle \right] \\ &+ O\left(\frac{1}{n^{\frac{d}{2}+1}}\right). \end{aligned}$$

Overcoming of problems

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- The term $\mathbb{E} \left\langle \frac{1}{n^d} \text{ex}_n(\eta_t^n - \rho_t^n) \tau(\eta_t^n - \rho_t^n), G(\eta_t^n) \right\rangle^2$ can be controlled via 4-point correlation function

$$\mathbb{E} \left[\prod_{i=1}^4 (\eta_t^n(x_i) - \rho_t^n(x_i)) \right] \lesssim \frac{1}{n^2}$$

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All computations and estimates for $(\mathcal{G}^n - \mathcal{G})F(\hat{\rho}, \hat{\zeta})$ can be easily transferred to the case $F \in C^{1,3}(H_J \times H_{-I})$.

Differentiability of $U_t(\rho, \zeta) = \mathbb{E}_{\rho, \zeta} F(\rho_t, \zeta_t)$

A solution to

$$\begin{aligned}d\rho_t &= \frac{1}{2} \Delta \rho_t dt \\d\zeta_t &= \frac{1}{2} \Delta \zeta_t dt + \nabla \cdot \left(\sqrt{\rho_t(1 - \rho_t)} dW_t \right)\end{aligned}$$

exists for all $\rho_0 \in L_2(\mathbb{T}^d; [0, 1])$ and $\zeta_0 \in H_{-I}$ for $I > \frac{d}{2} + 1$.

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Proposition [Gess, K. '24]

Let $I > \frac{d}{2} + 1$ and $F \in C_b^{2,4}(H_{-I})$. Then $U_t(\rho, \zeta) = \mathbb{E} F(\zeta_t) \in C_b^{1,3}(H_J \times H_{-I})$ for $J > \frac{d}{2}$. Moreover,

$$D_\rho U_t(\rho, \zeta) = \frac{1}{2} \mathbb{E} [D^2 F(\zeta_t) : DV_t(\rho)]$$

with

$$V_t(\rho)(\varphi, \varphi) = \text{Var} \langle \varphi, \zeta_t \rangle = \frac{1}{2} \int_0^t \langle |\nabla \varphi_{t-s}|^2, \rho_s (1 - \rho_s) \rangle ds$$

and φ_t is a solution to the heat equation started from φ .

Integration by parts

Let $\xi^z \sim N(m, z)$ for $z \geq 0$

$$\begin{aligned}\frac{d}{dz} \mathbb{E}[f(\xi^z)] &= \int_{\mathbb{R}} \frac{\partial}{\partial z} f(m + \sqrt{z}y) \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy \\ &= \int_{\mathbb{R}} f'(m + \sqrt{z}y) \frac{y}{2\sqrt{2\pi}z} e^{-\frac{y^2}{2}} dy \\ &= \frac{1}{2} \int_{\mathbb{R}} f''(m + \sqrt{z}y) \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy = \frac{1}{2} \mathbb{E}[f''(\xi^z)]\end{aligned}$$

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In particular,

$$\frac{d}{dz} \mathbb{E} [f(\xi^{V(z)})] = \frac{1}{2} \mathbb{E} [f''(\xi^z) V'(z)]$$

Idea of proof

Recall that $U_t(\rho) = \mathbb{E}F(\zeta_t)$. We show the differentiability of

$$U_t^n(\rho) := \mathbb{E}F(\text{pr}_n \zeta_t) = \mathbb{E}f_n(\xi_n),$$

where

$$\text{pr}_n \zeta_t = \sum_{k \in \mathbb{Z}_n^d} \langle \zeta_t, s_k \rangle s_k$$

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We show that

$$\begin{aligned} DU_t^n(\rho) &= \frac{1}{2} \mathbb{E} [D^2 f_n(\xi_n) : DV^n(\rho)] \\ &= \frac{1}{2} \mathbb{E} [D^2 F(\text{pr}_n \zeta_t) : \text{pr}_n DV_t(\rho)] \\ &\rightarrow \frac{1}{2} \mathbb{E} [D^2 F(\zeta_t) : DV_t(\rho)] \end{aligned}$$

Comparison of initial fluctuations

Quantitative CLT for SSEP

Berry-Esseen bound for the initial fluctuations

- It remains only to compare

$$\mathbb{E}U_t(\text{ex}_n\rho_0^n, \text{ex}_n\zeta_0^n) - \mathbb{E}U_t(\rho_0, \zeta_0)$$

where $\zeta_0^n = n^{d/2}(\eta_0^n - \rho_0^n)$, $\rho_0^n = \rho_0|_{\mathbb{T}_n^d}$, $\eta_0^n(x) \sim \text{B}(\rho_0(x))$, $x \in \mathbb{T}_n^d$,
i.i.d. and ζ_0 is a centered Gaussian distribution with

$$\text{Var} \langle \zeta_0, \varphi \rangle = \langle \rho_0(1 - \rho_0)\varphi, \varphi \rangle.$$

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where $G \in C^3(H_{-I})$, where

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- Is enough to compare for $g \in C^3(\mathbb{R}^{\mathbb{Z}_n^d})$

$$\mathbb{E}g\left(\left((1 + |k|^2)^{-I/2} \langle \zeta_0^n, s_k \rangle_n\right)_{k \in \mathbb{Z}_n^d}\right) - \mathbb{E}g\left(\left((1 + |k|^2)^{-I/2} \langle \zeta_0, s_k \rangle\right)_{k \in \mathbb{Z}_n^d}\right).$$

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- Apply multidimensional Berry-Essen theorem [e.g., Meckes '09]

Stein's method

Let (X, X') be an exchangeable pair of random variables in $\mathbb{R}^m$, i.e. $\text{Law}(X, X') = \text{Law}(X', X)$, such that

$$\mathbb{E}[X' - X|X] = -\Lambda X$$

and

$$\mathbb{E}[(X' - X)(X' - X)^T] = 2\Lambda\Sigma + E'$$

for some invertible matrix Λ , a symmetric non-negative definite matrix Σ and a $\sigma(X)$ -measurable random matrix E' .

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Then for each $f \in C^3(\mathbb{R}^m)$ one has

$$\begin{aligned} |\mathbb{E}f(X) - \mathbb{E}f(\xi)| &\leq \|\Lambda^{-1}\|_{op} \left[\frac{1}{4} \|D^2 f\|_{C,HS} \mathbb{E} \|E'\|_{HS} \right. \\ &\quad \left. + \frac{1}{9} \|D^3 f\|_{C,op} \mathbb{E} |X' - X|^3 \right], \end{aligned}$$

where $\xi \sim N(0, \Sigma)$.

Construction of exchangeable pair

Let $\eta_0^n(x) \sim B(\rho(x))$, $x \in \mathbb{T}_n^d$, and $\zeta := \zeta_0^n = n^{d/2} (\eta_0^n - \rho_0^n)$. Set

$$X_k := \langle \zeta, \varsigma_k \rangle_n, \quad k \in \mathbb{Z}_n^d,$$

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$$\zeta'(x) := \zeta(x) \mathbb{I}_{\{\gamma \neq x\}} + \bar{\zeta}(x) \mathbb{I}_{\{\gamma = x\}},$$

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$$\mathbb{E}[X'_k - X_k | X] = -\frac{1}{n^d} X,$$

$$\mathbb{E}[(X'_k - X_k)(X'_l - X_l) | X] = \frac{2}{n^d} \Sigma_{k,l} + \frac{1}{n^d} E'_{k,l},$$

with

$$\Sigma_{k,l} = \langle \rho(1 - \rho) \varsigma_k, \varsigma_l \rangle$$

$$E'_{k,l} = 2 \left[\langle \rho_n(1 - \rho_n) \varsigma_k, \varsigma_l \rangle_n - \langle \rho(1 - \rho) \varsigma_k, \varsigma_l \rangle \right] + \left\langle \left((\eta_n - \rho_n)^2 - \rho_n(1 - \rho_n) \right) \varsigma_k, \varsigma_l \right\rangle.$$

References

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Thank you!