A Quantitative Central Limit Theorem for Simple Symmetric Exclusion Process

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Theory of Markov Semigroups and Schrödinger Operators – Wrocław

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DER FORSCHUNG | DER LEHRE | DER BILDUNG

LLN and CLT

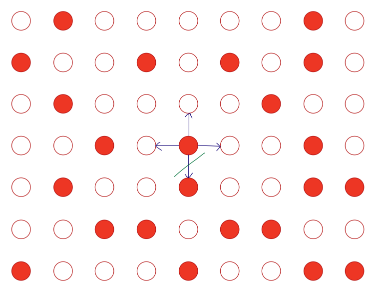
Quantitative CLT for SSEP

Simple symmetric exclusion process

On the d-dim discrete torus

$$\mathbb{T}_n^d := \left\{ \frac{k}{n} : \ k \in \mathbb{Z}_n^d := \{0, \dots, n-1\}^d \right\} \subset \mathbb{T}^d = (\mathbb{R}/\mathbb{Z})^d$$

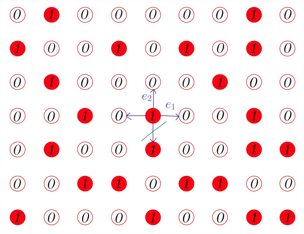
we consider a **Simple Symmetric Exclusion Process** (SSEP)



State space and generator

Particle configuration $\eta \in \{0,1\}^{\mathbb{T}_n^d}$:

$$\eta(x) = 0 \Leftrightarrow \text{side } x \text{ is empty}$$
$$\eta(x) = 1 \Leftrightarrow \text{side } x \text{ is occupied}$$



$$\eta^{x\leftrightarrow y}(z) = \begin{cases} \eta(z), & z \neq x, y, \\ \eta(y), & z = x, \\ \eta(x), & z = y, \end{cases}$$

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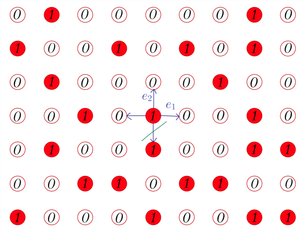

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$$\mathcal{G}^n F(\eta) := \frac{n^2}{2} \sum_{j=1}^{d} \sum_{x \in \mathbb{T}_n} \left[F(\eta^{x\leftrightarrow x+e_j}) - F(\eta) \right]$$ [Kipnis, Landim '99]

SSEP is already parabolically rescaled: space $\sim \frac{1}{n}$ time $\sim n^2$!

Non-equilibrium SSEP

Let η_t^n, $t \geq 0$, be a SSEP and $\rho_0 : \mathbb{T}^d \to [0, 1]$ be an initial profile.

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The process $\rho_t^n(x) := \mathbb{E}\eta_t^n(x)$ solves the discrete stochastic Heat equation

$$d\rho_t^n(x) = \frac{1}{2}\Delta_n \rho_t^n(x) dt$$

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Thus,

$$\rho_t^n := \frac{1}{n^d}\sum_{x\in\mathbb{T}_n^d} \rho_t^n(x)\delta_x \to \rho_t,$$

where ρ_t solves

$$d\rho_t = \frac{1}{2}\Delta\rho_t dt$$

with initial condition ρ_0.

Law of large numbers

Theorem **[see e.g. in Kipnis, Landim '99]**

Let $\rho_0 : \mathbb{T}^d \to [0,1]$ be an initial density profile and $\eta_0^n(x) \sim \mathrm{Bernulli}(\rho_0(x))$ be independent. Then

$$\eta_t^n := \frac{1}{n^d} \sum_{x \in \mathbb{T}_n^d} \eta_t^n(x) \delta_x, \quad t \geq 0$$

converges in probability to $\rho_t(x)$, $t \geq 0$, where ρ_t solves

$$d\rho_t = \frac{1}{2} \Delta \rho_t dt$$

with initial condition ρ_0.

Convergence of generator

Note that $\langle \varphi, \eta_t^n \rangle$ solves the martingale problem

$$f\left(\langle \varphi, \eta_t^n \rangle\right) - \int_0^t \mathcal{G}_n^{EP} f\left(\langle \varphi, \eta_s^n \rangle\right) ds \quad \text{is a mart.,}$$

where

$$\begin{aligned}
\mathcal{G}^n f\left(\langle \varphi, \eta \rangle\right) :=& \frac{n^2}{2} \sum_{j=1}^{d} \sum_{x \in \mathbb{T}_n} f'\left(\langle \varphi, \eta \rangle\right) \quad \underbrace{\left(\left\langle \varphi, \eta^{x \leftrightarrow x+e_j} \right\rangle - \langle \varphi, \eta \rangle\right)}_{\frac{1}{n^d}\left[\varphi(x+e_j)-\varphi(x)\right]\left[\eta(x)-\eta(x+e_j)\right]} \\
&+ \frac{n^2}{4} \sum_{j=1}^{d} \sum_{x \in \mathbb{T}_n} f''\left(\langle \varphi, \eta \rangle\right) \left(\left\langle \varphi, \eta^{x \leftrightarrow x+e_j} \right\rangle - \langle \varphi, \eta \rangle\right)^2 \\
=& \frac{1}{2} f'\left(\langle \varphi, \eta \rangle\right) \langle \Delta_n \varphi, \eta \rangle \\
&+ \frac{1}{4n^d} f''\left(\langle \varphi, \eta \rangle\right) \sum_{j=1}^{d} \left\langle \left|\partial_{n,j} \varphi\right|^2, \tau_j \eta + \eta - 2\eta \tau_j \eta \right\rangle + O\left(\frac{\|f'''\|}{n^{2d+1}}\right)
\end{aligned}$$

Density fluctuation field and CLT

We now consider the fluctuations of the SSEP around its mean:

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The generator of ζ_t^n can be expanded as follows

$$\begin{aligned}\mathcal{G}^n f\left(\langle \varphi, \zeta \rangle\right) &= \frac{1}{2} f'\left(\langle \varphi, \zeta \rangle\right) \langle \Delta_n \varphi, \tilde{\zeta} \rangle + \frac{n^d}{4n^d} f''\left(\langle \varphi, \zeta \rangle\right) \left\langle \left| \nabla_n \varphi \right|^2, \tau\eta + \eta - 2\eta\tau\eta \right\rangle \\ &\quad + O\left(1/n^{\frac{d}{2}+1}\right) \\ \xrightarrow[n\to\infty]{} \\ \mathcal{G} f\left(\langle \varphi, \zeta \rangle\right) &= \frac{1}{2} f'\left(\langle \varphi, \zeta \rangle\right) \langle \Delta \varphi, \zeta \rangle + \frac{1}{4} f''\left(\langle \varphi, \zeta \rangle\right) \left\langle \left| \nabla \varphi \right|^2, \rho + \rho - 2\rho^2 \right\rangle\end{aligned}$$

Central limit theorem

Theorem 2 **[Galves, Kipnis, Spohn; Ravishankar '90]**

Let the initial density profile ρ_0 be smooth. Then the density fluctuation field

$$\zeta_t^n := \frac{1}{n^d} \sum_{x \in \mathbb{T}_n^d} \zeta_t(x) \delta_x$$

converges in $D\left([0,T], \mathcal{D}'\right)$ to the generalized Ornstein-Uhlenbeck process that solves the linear SPDE

$$d\zeta_t = \frac{1}{2} \Delta \zeta_t dt + \nabla \cdot \left(\sqrt{\rho_t (1 - \rho_t)} dW_t \right)$$

with the centered Gaussian initial condition such that

$$\mathbb{E}\left[\langle \zeta_0, \varphi \rangle^2 \right] = \langle \rho_0 (1 - \rho_0) \varphi, \varphi \rangle$$

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- [Chassagneux, Szpruch, Tse '22]: Weak quantitative propagation of chaos
  (mean field limit)
- [Kolokoltsov '10] Central limit theorem for the Smoluchovski coagulation model
  (mean field limit, non-local Smoluchowski's coagulation equation)
- ...

Main result

Theorem 3 **[Gess, K. '24]**

Let

- the initial density profile $\rho_0 : \mathbb{T}^d \to [0,1]$ be smooth enough,
- η_t^n be SSEP with $\eta_0^n(x) \sim \mathrm{Bernulli}(\rho_0(x))$ and independent.

Then

$$\sup_{t\in[0,T]} \left|\mathbb{E} f\left(\langle \vec{\varphi}, \zeta_t^n \rangle\right) - \mathbb{E} f\left(\langle \vec{\varphi}, \zeta_t \rangle\right)\right| \leq \frac{C}{n^{\frac{d}{2}\wedge 1}} \|f\|_{\mathrm{C}_l^3} \|\vec{\varphi}\|_{\mathrm{C}^I}$$

for all $n \geq 1$, $f \in \mathrm{C}_b^3(\mathbb{R}^m)$ and $\vec{\varphi} \in \left(\mathrm{C}^I(\mathbb{T}^d)\right)^m$, where I is large enough.

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The rate $\frac{1}{n^{\frac{d}{2}\wedge 1}}$ is optimal:

$\frac{1}{n}$ – lattice discretization error, $\frac{1}{n^{\frac{d}{2}}}$ – particle approximation error

Main tool

Idea of proof: Compare two (time-homogeneous) Markov processes X_t, Y_t using

$$\mathbb{E}F(X_t) - \mathbb{E}F(Y_t) = \mathbb{E}U_t(X_0) - \mathbb{E}U_t(Y_0) \\ + \int_0^t \mathbb{E}\left[\left(\mathcal{G}^X - \mathcal{G}^Y\right) U_{t-s}(X_s)\right] ds,$$

where $U_t(x) = \mathbb{E}_x F(Y_t)$.

[see e.g. Ethier, Kurtz ’86]

Comparison of semigroup

We consider the Markov processes:

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- solution to heat equation and generalized OU process $Y = (\rho_t, \zeta_t)$.

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where $U_t(\rho, \zeta) = \mathbb{E}_{\rho,\zeta} F(\rho_t, \zeta_t)$

Comparison of dynamics

Quantitative CLT for SSEP

Generators

We start from the formal computation for cylindrical functions:

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$$\mathcal{G}^n U(\rho,\zeta) = \ldots + \frac{1}{4}\partial_2^2 f\left\langle |\nabla_n\varphi|^2, \tau\eta+\eta-2\eta\tau\eta\right\rangle + O\left(1/n^{\frac{d}{2}+1}\right),$$

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$$\begin{aligned} \mathcal{G}^n U(\rho, \zeta) &= \dots + \frac{1}{4}\partial_2^2 f \left\langle |\nabla_n \varphi|^2, \tau\eta + \eta - 2\eta\tau\eta \right\rangle + O\left(1/n^{\frac{d}{2}+1}\right), \\ \mathcal{G} U(\rho, \zeta) &= \dots + \frac{1}{4}\partial_2^2 f \left\langle |\nabla \varphi|^2, 2\rho - 2\rho^2 \right\rangle. \end{aligned}$$

Thus

$$\begin{aligned} \left(\mathcal{G}^n - \mathcal{G}\right) U(\rho, \zeta) = \dots + \frac{1}{4}\partial_2^2 f \left[\left\langle |\nabla_n \varphi|^2, \tau\eta + \eta - 2\eta\tau\eta \right\rangle - \left\langle |\nabla \varphi|^2, 2\rho - 2\rho^2 \right\rangle \right] \\ + O\left(\frac{1}{n^{\frac{d}{2}+1}}\right). \end{aligned}$$

The difficulties

$$\left\langle |\nabla_n \varphi|^2, \tau\eta + \eta - 2\eta\tau\eta \right\rangle - \left\langle |\nabla \varphi|^2, 2\rho - 2\rho^2 \right\rangle$$

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We need different lifting of the particle system to the Sobolev spaces.

Discrete and continuous Fourier transform

We will replace $\rho = \frac{1}{n^d} \sum_{x \in \mathbb{T}_n^d} \rho(x) \delta_x$ and $\zeta = \frac{1}{n^d} \sum_{x \in \mathbb{T}_n^d} \zeta(x) \delta_x$ by a smooth interpolation.

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- Let $L_2(\mathbb{T}_n^d)$ be the Hilbert space of all functions on $\mathbb{T}_n^d$ with inner product

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- $\varsigma_k(x) = e^{2\pi \mathrm{i} k\cdot x}$, $k\in\mathbb{Z}^d$, $x\in\mathbb{T}^d\supset\mathbb{T}_n^d$
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$$L_2(\mathbb{T}_n^d) \ni \rho = \sum_{k \in \mathbb{Z}_n^d} \langle \rho, \varsigma_k \rangle_n \varsigma_k \text{ on } \mathbb{T}_n^d, \quad L_2(\mathbb{T}^d) \ni g = \sum_{k \in \mathbb{Z}^d} \langle g, \varsigma_k \rangle_{L_2} \varsigma_k \text{ on } \mathbb{T}^d$$

New (smooth) lifting of discrete space

For functions $\rho \in L_2(\mathbb{T}_n^d)$ and $\varphi \in L_2(\mathbb{T}^d)$ define

$$\mathrm{ex}_n \rho := \sum_{k \in \mathbb{Z}_n^d} \langle \rho, \varsigma_k \rangle_n \varsigma_k \quad \text{on } \mathbb{T}^d, \quad \mathrm{pr}_n \varphi := \sum_{k \in \mathbb{Z}_n^d} \langle \varphi, \varsigma_k \rangle_{L_2} \varsigma_k \quad \text{on } \mathbb{T}^d$$

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Basic properties of $\mathrm{ex}_n f$ and $\mathrm{pr}_n g$

- $\mathrm{ex}_n \rho = \rho$ on $\mathbb{T}_n^d$ and $\mathrm{ex}_n \rho \in \mathrm{C}^\infty(\mathbb{T}^d)$
- $\mathrm{pr}_n \varphi$ is well defined on $\mathbb{T}_n^d$ for each $\varphi \in H_J$, $J \in \mathbb{R}$.
- $\langle \rho_1, \rho_2 \rangle_n = \langle \mathrm{ex}_n \rho_1, \mathrm{ex}_n \rho_2 \rangle_{L_2}$ and $\langle \rho, \mathrm{pr}_n g \rangle_n = \langle \mathrm{ex}_n \rho, g \rangle_{L_2}$
- $\|\mathrm{pr}_n g - g\|_{H_J} \leq \frac{1}{n} \|g\|_{H_{J+1}}$, $\|\mathrm{ex}_n \varphi - \varphi\|_{H_J} \leq \frac{C}{n} \|\varphi\|_{\mathrm{C}^{J+2+\frac{d}{2}}}$, ...

Comparison of generators for smooth interp.

We replace

$$\frac{1}{n^d}\sum_{x\in\mathbb{T}_n^d}\rho(x)\delta_x \rightsquigarrow \mathrm{ex}_n\rho =: \hat{\rho}, \qquad \frac{1}{n^d}\sum_{x\in\mathbb{T}_n^d}\zeta(x)\delta_x \rightsquigarrow \mathrm{ex}_n\zeta =: \hat{\zeta}$$

in particular

$$\langle\varphi, \mathrm{ex}_n\zeta\rangle_{L_2} = \langle \mathrm{pr}_n\varphi, \zeta\rangle_n = \langle \mathrm{pr}_n\varphi, \zeta\rangle$$

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Thus

$$\begin{aligned}(\mathcal{G}^n - \mathcal{G})F(\mathrm{ex}_n\rho, \mathrm{ex}_n\rho) &= \frac{1}{2}\partial_1 f[...] \\ &+ \frac{1}{4}\partial_2^2 f\left[\left\langle \mathrm{ex}_n\left|\nabla_n \mathrm{pr}_n\varphi\right|^2, \widehat{\tau\eta} + \hat{\eta} - 2\mathrm{ex}_n(\eta\tau\eta)\right\rangle - \left\langle\left|\nabla\varphi\right|^2, 2\hat{\rho} - 2\hat{\rho}^2\right\rangle\right] \\ &+ O\left(\frac{1}{n^{\frac{d}{2}+1}}\right).\end{aligned}$$

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We deal with the most problematic term $\mathrm{ex}_n(\eta\tau\eta) - \hat{\rho}^2$ as follows

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- $\|\mathrm{ex}_n(\rho\tau\zeta)\| \leq \|\hat{\rho}\|_{C^J}\|\hat{\zeta}\|_{H_{-I}}$.
- The term $\mathbb{E}\left\langle \frac{1}{n^d}\,\mathrm{ex}_n\left(\eta_t^n - \rho_t^n\right)\tau\left(\eta_t^n - \rho_t^n\right), G(\eta_t^n)\right\rangle^2$ can be controlled via 4-point correlation function

$$\mathbb{E}\left[\prod_{i=1}^{4}\left(\eta_t^n(x_i) - \rho_t^n(x_i)\right)\right] \lesssim \frac{1}{n^2}$$

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All computations and estimates for $(\mathcal{G}^n - \mathcal{G})\,F(\hat{\rho},\hat{\zeta})$ can be easily transferred to the case $F \in \mathrm{C}^{1,3}(H_J \times H_{-I})$.

Differentiability of $U_t(\rho, \zeta) = \mathbb{E}_{\rho,\zeta} F(\rho_t, \zeta_t)$

A solution to

$$\begin{aligned} d\rho_t &= \frac{1}{2}\Delta \rho_t dt \\ d\zeta_t &= \frac{1}{2}\Delta \zeta_t dt + \nabla \cdot \left(\sqrt{\rho_t(1-\rho_t)} dW_t \right) \end{aligned}$$

exists for all $\rho_0 \in L_2(\mathbb{T}^d; [0,1])$ and $\zeta_0 \in H_{-I}$ for $I > \frac{d}{2} + 1$.

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Proposition **[Gess, K. '24]**

Let $I > \frac{d}{2}+1$ and $F \in \mathrm{C}_b^{2,4}(H_{-I})$. Then $U_t(\rho,\zeta) = \mathbb{E}F(\zeta_t) \in \mathrm{C}_b^{1,3}(H_J \times H_{-I})$ for $J > \frac{d}{2}$. Moreover,

$$
D_1 U_t(\rho,\zeta) = \frac{1}{2}\mathbb{E}\left[D^2 F(\zeta_t) : DV_t(\rho)\right]
$$

with

$$
V_t(\rho)(\varphi,\varphi) = \mathrm{Var}\,\langle\varphi,\zeta_t\rangle = \frac{1}{2}\int_0^t \left\langle |\nabla\varphi_{t-s}|^2, \rho_s(1-\rho_s)\right\rangle ds
$$

and φ_t is a solution to the heat equation started from φ.

Integration by parts

Let $\xi^z \sim N(m, z)$ for $z \geq 0$

$$\begin{aligned}
\frac{d}{dz}\mathbb{E}\left[f\left(\xi^z\right)\right] &= \int_{\mathbb{R}} \frac{\partial}{\partial z} f\left(m+\sqrt{z}y\right)\frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}\,dy \\
&= \int_{\mathbb{R}} f'\left(m+\sqrt{z}y\right)\frac{y}{2\sqrt{2\pi z}}e^{-\frac{y^2}{2}}\,dy \\
&= \frac{1}{2}\int_{\mathbb{R}} f''\left(m+\sqrt{z}y\right)\frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}\,dy = \frac{1}{2}\mathbb{E}\left[f''\left(\xi^z\right)\right]
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\end{aligned}$$

In particular,

$$\frac{d}{dz}\mathbb{E}\left[f\left(\xi^{V(z)}\right)\right] = \frac{1}{2}\left[f''\left(\xi^z\right)V'(z)\right]$$

Idea of proof

Recall that $U_t(\rho) = \mathbb{E}F(\zeta_t)$. We show the differentiability of

$$U_t^n(\rho) := \mathbb{E}F(\mathrm{pr}_n \zeta_t) = \mathbb{E}f_n(\xi_n),$$

where

$$\mathrm{pr}_n \zeta_t = \sum_{k \in \mathbb{Z}_n^d} \langle \zeta_0, \varsigma_k \rangle \varsigma_k$$

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We show that

$$\begin{aligned} DU_t^n(\rho) &= \frac{1}{2}\mathbb{E}\left[D^2 f_n(\xi_n) : DV^n(\rho)\right] \\ &= \frac{1}{2}\mathbb{E}\left[D^2 F(\mathrm{pr}_n\zeta_t) : \mathrm{pr}_n DV_t(\rho)\right] \\ &\to \frac{1}{2}\mathbb{E}\left[D^2 F(\zeta_t) : DV_t(\rho)\right] \end{aligned}$$

Comparison of initial fluctuations

Quantitative CLT for SSEP

Berry-Esseen bound for the initial fluctuations

- It remains only to compare

$$\mathbb{E} U_t(\mathrm{ex}_n \rho_0^n, \mathrm{ex}_n \zeta_0^n) - \mathbb{E} U_t(\rho_0, \zeta_0)$$

where $\zeta_0^n = n^{d/2}\left(\eta_0^n - \rho_0^n\right)$, $\rho_0^n = \rho_0\big|_{\mathbb{T}_n^d}$, $\eta_0^n(x) \sim \mathrm{Bernulli}\left(\rho_0(x)\right)$, $x \in \mathbb{T}_n^d$, i.i.d. and ζ_0 is a centered Gaussian distribution with

$$\mathrm{Var}\,\langle \zeta_0, \varphi \rangle = \langle \rho_0(1-\rho_0)\varphi, \varphi \rangle.$$

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- It is enough to compare only
$$\mathbb{E}G(\mathrm{ex}_n\zeta_0^n) - \mathbb{E}G(\mathrm{pr}_n\zeta_0),$$
where $G \in \mathrm{C}^3(H_{-I})$, where
$$\mathrm{ex}_n\zeta_0^n := \sum_{k\in\mathbb{Z}_n^d} \langle \zeta_0^n, \varsigma_k\rangle_n \varsigma_k, \quad \mathrm{pr}_n\zeta_0 := \sum_{k\in\mathbb{Z}_n^d} \langle \zeta_0, \varsigma_k\rangle \varsigma_k$$

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- Is enough to compare for $g \in \mathrm{C}^3\left(\mathbb{R}^{\mathbb{Z}_n^d}\right)$
$$\mathbb{E}g\left(\left((1+|k|^2)^{-I/2}\langle \zeta_0^n, \varsigma_k \rangle_n\right)_{k\in\mathbb{Z}_n^d}\right) - \mathbb{E}g\left(\left((1+|k|^2)^{-I/2}\langle \zeta_0, \varsigma_k \rangle\right)_{k\in\mathbb{Z}_n^d}\right).$$

Berry-Esseen bound for the initial fluctuations

- It remains only to compare

$$\mathbb{E}U_t(\mathrm{ex}_n\rho_0^n, \mathrm{ex}_n\zeta_0^n) - \mathbb{E}U_t(\rho_0, \zeta_0)$$

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- Apply multidimensional Berry-Essen theorem [e.g., Meckes ’09]

Stein's method

Let (X, X') be an exchangeable pair of random variables in $\mathbb{R}^m$, i.e. $\mathrm{Law}(X, X') = \mathrm{Law}(X', X)$, such that

$$\mathbb{E}[X' - X|X] = -\Lambda X$$

and

$$\mathbb{E}\left[(X' - X)(X' - X)^T\right] = 2\Lambda\Sigma + E'$$

for some invertible matrix Λ, a symmetric non-negative definite matrix Σ and a $\sigma(X)$-measurable random matrix E'.

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Then for each $f \in \mathrm{C}^3(\mathbb{R}^m)$ one has

$$\begin{aligned}|\mathbb{E}f(X) - \mathbb{E}f(\xi)| \leq \left\|\Lambda^{-1}\right\|_{op} \Bigg[\frac{1}{4}\left\|D^2 f\right\|_{C,HS} \mathbb{E}\left\|E'\right\|_{HS} \\ + \frac{1}{9}\left\|D^3 f\right\|_{C,op} \mathbb{E}\left|X' - X\right|^3\Bigg],\end{aligned}$$

where $\xi \sim N(0, \Sigma)$.

Construction of exchangeable pair

Let $\eta_0^n(x) \sim \mathrm{Bernulli}(\rho(x))$, $x \in \mathbb{T}_n^d$, and $\zeta := \zeta_0^n = n^{d/2}\left(\eta_0^n - \rho_0^n\right)$. Set

$$X_k := \langle \zeta, \varsigma_k \rangle_n, \quad k \in \mathbb{Z}_n^d,$$

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$$\zeta'(x) := \zeta(x)\mathbb{I}_{\{\gamma \neq x\}} + \bar{\zeta}(x)\mathbb{I}_{\{\gamma = x\}},$$

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$$X_k' := \langle \zeta', \varsigma_k \rangle_n, \quad k \in \mathbb{Z}_n^d,$$

$$\mathbb{E}\left[X_k' - X_k | X\right] = -\frac{1}{n^d} X,$$

$$\mathbb{E}\left[\left(X_k' - X_k\right)\left(X_l' - X_l\right) | X\right] = \frac{2}{n^d}\Sigma_{k,l} + \frac{1}{n^d} E_{k,l}',$$

with

$$\Sigma_{k,l} = \langle \rho(1-\rho)\varsigma_k, \varsigma_l \rangle$$

$$E_{k,l}' = 2\left[\langle \rho_n(1-\rho_n)\varsigma_k, \varsigma_l \rangle_n - \langle \rho(1-\rho)\varsigma_k, \varsigma_l \rangle\right] + \left\langle \left((\eta_n - \rho_n)^2 - \rho_n(1-\rho_n)\right)\varsigma_k, \varsigma_l \right\rangle$$

References

[1] Benjamin Gess and Vitalii Konarovskyi.
A quantitative central limit theorem for the simple symmetric exclusion process (2024),
arXiv:2408.01238

Thank you!