

Higher-Order Gaussian Approximations for Simple Symmetric Exclusion Process

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Stochastic Analysis in the Sciences

June 24, 2026



Universität Hamburg

DER FORSCHUNG | DER LEHRE | DER BILDUNG

LLN and CLT

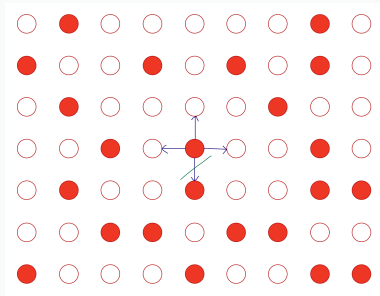
Quantitative CLT for SSEP

Simple symmetric exclusion process

On the d -dim discrete torus

$$\mathbb{T}_n^d := \left\{ \frac{k}{n} : k \in \{0, \dots, n-1\}^d \right\} \subset \mathbb{T}^d = (\mathbb{R}/\mathbb{Z})^d$$

we consider a **Simple Symmetric Exclusion Process** (SSEP)

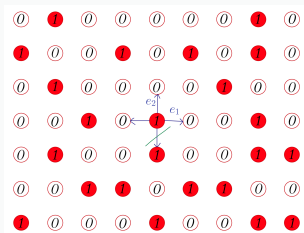


State space and generator

Particle configuration $\eta \in \{0, 1\}^{\mathbb{T}_n^d}$:

$\eta(x) = 0 \Leftrightarrow$ side x is empty

$\eta(x) = 1 \Leftrightarrow$ side x is occupied



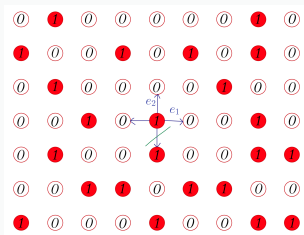
$$\eta^{x \leftrightarrow y}(z) = \begin{cases} \eta(z), & z \neq x, y, \\ \eta(y), & z = x, \\ \eta(x), & z = y, \end{cases}$$

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$$\eta^{x \leftrightarrow y}(z) = \begin{cases} \eta(z), & z \neq x, y, \\ \eta(y), & z = x, \\ \eta(x), & z = y, \end{cases}$$

$$\mathcal{L}^n F(\eta) := \frac{n^2}{2} \sum_{j=1}^d \sum_{x \in \mathbb{T}_n} [F(\eta^{x \leftrightarrow x+e_j}) - F(\eta)] \quad [\text{Kipnis, Landim '99}]$$

SSEP is already parabolically rescaled: space $\sim \frac{1}{n}$ time $\sim n^2$!

Local equilibrium SSEP

Let $\eta_t^n, t \geq 0$, be a SSEP and $\rho^0 : \mathbb{T}^d \rightarrow [0, 1]$ be an initial profile.

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The process $\rho_t^n(x) := \mathbb{E}\eta_t^n(x)$ solves the discrete stochastic Heat equation

$$\partial_t \rho_t^n(x) = \frac{1}{2} \Delta_n \rho_t^n(x)$$

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Thus,

$$\rho_t^n := \frac{1}{n^d} \sum_{x \in \mathbb{T}_n^d} \rho_t^n(x) \delta_x \rightarrow \varrho_t,$$

where ϱ_t solves

$$\partial_t \varrho_t = \frac{1}{2} \Delta \varrho_t$$

with initial condition ρ^0 .

Law of large numbers

Theorem [see e.g. in Kipnis, Landim '99]

Let $\rho^0 : \mathbb{T}^d \rightarrow [0, 1]$ be an initial density profile and $\eta_0^n(x) \sim \text{Bernulli}(\rho_0(x))$ be independent. Then

$$\eta_t^n := \frac{1}{n^d} \sum_{x \in \mathbb{T}_n^d} \eta_t^n(x) \delta_x, \quad t \geq 0$$

converges in probability to $\varrho_t(x)$, $t \geq 0$, where ϱ_t solves

$$\partial_t \varrho_t = \frac{1}{2} \Delta \varrho_t$$

with initial condition ρ^0 .

Central limit theorem

We now consider the fluctuations of the SSEP around its mean:

$$\zeta_t^n := n^{\frac{d}{2}} (\eta_t^n - \rho_t^n).$$

Theorem [Galves, Kipnis, Spohn; Ravishankar '90]

Let the initial density profile ρ_0 be smooth. Then the density fluctuation field

$$\bar{\zeta}_t^n := \frac{1}{n^d} \sum_{x \in \mathbb{T}_n^d} \zeta_t^n(x) \delta_x$$

converges in $D([0, T], \mathcal{D}')$ to the generalized Ornstein-Uhlenbeck process that solves the linear SPDE

$$\partial_t \xi_t = \frac{1}{2} \Delta \xi_t + \nabla \cdot \left(\sqrt{\varrho_t(1 - \varrho_t)} d\dot{W}_t \right)$$

with the centered Gaussian initial condition such that

$$\mathbb{E} [\langle \xi_0, \varphi \rangle^2] = \langle \rho^0(1 - \rho^0) \varphi, \varphi \rangle$$

Quantitative CLT and discretization error

Theorem [Gess, K. '24]

Let

- the initial density profile $\rho^0 : \mathbb{T}^d \rightarrow [0, 1]$ be smooth enough,
- η_t^n be SSEP with $\eta_0^n(x) \sim \text{Bernulli}(\rho^0(x))$ and independent.

Then

$$\sup_{t \in [0, T]} \left| \mathbb{E} f(\langle \varphi, \bar{\zeta}_t^n \rangle) - \mathbb{E} f(\langle \varphi, \xi_t \rangle) \right| \leq \frac{C}{n^{\frac{d}{2} \wedge 1}} \|f\|_{C_t^3} \|\varphi\|_{C^I}$$

for all $n \geq 1$, $f \in C_b^3(\mathbb{R}^m)$ and $\varphi \in C^I(\mathbb{T}^d)$, where I is large enough.

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The rate $\frac{1}{n^{\frac{d}{2} \wedge 1}}$ is optimal:

$\frac{1}{n}$ – lattice discretization error, $\frac{1}{n^{\frac{d}{2}}}$ – particle approximation error

Main tool

Idea of proof: Compare two Markov processes X_t, Y_t using

$$\begin{aligned}\mathbb{E}F(X_t) - \mathbb{E}F(Y_t) &= \mathbb{E}U_t(X_0) - \mathbb{E}U_t(Y_0) \\ &\quad + \int_0^t \mathbb{E} \left[(\mathcal{G}_s^X - \mathcal{G}_s^Y) U_{s,t}(X_s) \right] ds,\end{aligned}$$

where $U_{s,t}(x) = \mathbb{E} [F(Y_{s,t}(x))]$.

[see e.g. Ethier, Kurtz '86]

Comparison of dynamics

Quantitative CLT for SSEP

Comparison of generators (old approach)

For homogeneous Markov process $X_t = (\text{ex}_n \zeta_t, \text{ex}_n \rho_t) = (\hat{\zeta}_t, \hat{\rho}_t)$,

where $\text{ex}_n = \hat{\cdot}$ is a smooth (Fourier) interpolatoin

$$\begin{aligned} \mathcal{L}^n f(\langle \varphi, \hat{\zeta} \rangle, \langle \psi, \hat{\rho} \rangle) &= \frac{1}{2} f'_1(\langle \varphi, \hat{\zeta} \rangle, \langle \psi, \hat{\rho} \rangle) \langle \Delta_n \text{pr}_n \varphi, \zeta \rangle_n + \frac{1}{2} f'_2(\dots) \langle \Delta_n \text{pr}_n \psi, \rho \rangle_n \\ &\quad + \frac{1}{4} f''_2(\dots) \langle |\nabla_n \text{pr}_n \varphi|^2, \tau \eta + \eta - 2\eta \tau \eta \rangle_n + O\left(1/n^{\frac{d}{2}+1}\right) \end{aligned}$$

$$\begin{aligned} \mathcal{G} f(\langle \varphi, \hat{\zeta} \rangle, \langle \psi, \hat{\rho} \rangle) &= \frac{1}{2} f'_1(\langle \varphi, \hat{\zeta} \rangle, \langle \psi, \hat{\rho} \rangle) \langle \Delta \varphi, \hat{\zeta} \rangle + \frac{1}{2} f'_2(\dots) \langle \Delta \psi, \hat{\rho} \rangle \\ &\quad + \frac{1}{4} f''_2(\dots) \langle |\nabla \varphi|^2, \hat{\rho} + \hat{\rho} - 2(\hat{\rho})^2 \rangle \end{aligned}$$

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$$\langle \Delta_n \text{pr}_n \varphi, \zeta \rangle_n - \langle \Delta \varphi, \hat{\zeta} \rangle = O\left(\frac{1}{n}\right), \langle |\nabla_n \text{pr}_n \varphi|^2, \zeta \rangle_n - \langle |\nabla \varphi|^2, \hat{\rho} \rangle = O\left(\frac{1}{n}\right), \dots$$

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Problem: Many errors are mixed.

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Problem: Many errors are mixed.

Goal: Split errors and propose better approximation

Removing pr_n and ρ from comparison

Consider

$$\bar{\zeta}_t := \frac{1}{n^d} \sum_{x \in \mathbb{T}_n^d} \zeta_t(x) \delta_x = \frac{1}{n^d} \sum_{x \in \mathbb{T}_n^d} n^{\frac{d}{2}} (\eta_t(x) - \rho_t(x)) \delta_x$$

as non-homogeneous Markov process with generator $\mathcal{L}_{\rho_t}^n$.

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Lemma

Let $I > \frac{d}{2}$ and let $F \in C^3(H_{-I})$ be such that

$$D^2 F(f; \cdot) = D^2 F(f) [\delta^{\otimes 2}] \in C((\mathbb{T}^d)^2) \quad \text{and} \quad D^3 F(f; \cdot) \in C((\mathbb{T}^d)^3),$$

Then $\forall \eta \in \{0, 1\}^{\mathbb{T}_n^d}$, $\forall \rho \in [0, 1]^{\mathbb{T}_n^d}$ and $\zeta = n^{d/2}(\eta - \rho)$

$$\left| \mathcal{L}_{\rho}^n F(\bar{\zeta}) - \frac{1}{2} \langle \Delta_n D F(\bar{\zeta}), \bar{\zeta} \rangle - \frac{1}{4} \sum_{j=1}^d \langle \partial_{n,j}^{\odot 2} D^2 F(\bar{\zeta}), \overline{(\tau_{n,j} \eta - \eta)^2} \rangle \right|$$

$$\leq \frac{d}{12n^{\frac{d}{2}+1}} \sup_{f \in H_{-I}} \|D^3 F(f)\|_{C^3((\mathbb{T}^d)^3)}$$

Particle generator vs. OU generator

For $\zeta = n^{d/2}(\eta - \rho)$

Particle generator:

$$\mathcal{L}_\rho^n F(\bar{\zeta}) = \frac{1}{2} \langle \Delta_n D F(\bar{\zeta}), \bar{\zeta} \rangle + \frac{1}{4} \sum_{j=1}^d \langle \partial_{n,j}^{\odot 2} D^2 F(\bar{\zeta}), (\tau_{n,j} \eta - \eta)^2 \rangle_n + O\left(\frac{1}{n^{d/2+1}}\right)$$

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Recall that Ornstein-Uhlenbeck process is defined by

$$d\xi_t = \frac{1}{2} \Delta \xi_t dt + \nabla \cdot \left(\sqrt{\varrho_t(1 - \varrho_t)} dW_t \right)$$

Ornstein–Uhlenbeck generator

$$\mathcal{G}_\varrho F(\bar{\zeta}) = \frac{1}{2} \langle \Delta D F(\bar{\zeta}), \bar{\zeta} \rangle + \frac{1}{2} \sum_{j=1}^d \langle \partial_j^{\odot 2} D^2 F(\bar{\zeta}), \varrho_t(1 - \varrho_t) \rangle$$

Errors

$$\mathcal{L}_\rho^n F(\bar{\zeta}) = \frac{1}{2} \langle \Delta_n D F(\bar{\zeta}), \bar{\zeta} \rangle + \frac{1}{4} \sum_{j=1}^d \langle \partial_{n,j}^{\odot 2} D^2 F(\bar{\zeta}), (\tau_{n,j} \eta - \eta)^2 \rangle_n + O\left(\frac{1}{n^{d/2+1}}\right)$$

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Errors:

- $\Delta_n \varphi - \Delta \varphi$ in H_I
- $\partial_n \varphi - \partial_j \varphi$ in H_I
- $\langle \varphi, 1 \rangle_n - \langle \varphi, 1 \rangle$
- $(\tau_{n,j} \eta - \eta)^2 - 2\varrho_t(1 - \varrho_t)$

Approximation of discrete operators

Using Taylor's formula, we get

$$\begin{aligned}\partial_{n,j}\varphi &= n \left(\varphi(\cdot + e_j^n) - \varphi \right) \\ &= \sum_{k=1}^m \frac{\partial_j^k}{k!n^{k-1}} \varphi + O\left(\frac{1}{n^{m+1}} \|\partial_j^{m+1}\varphi\|_{C^l(\mathbb{T}^d)}\right)\end{aligned}$$

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Set

$$\tau_{n,j}^{(m)} := \sum_{k=0}^m \frac{\partial_j^k}{k!n^k}, \quad \partial_{n,j}^{(m)} = \sum_{k=1}^m \frac{\partial_j^k}{k!n^{k-1}}, \quad \Delta_n^{(2m)} = \sum_{k=1}^m \frac{1}{(2k)!n^{2k-2}} \sum_{j=1}^d \partial_j^{2k}.$$

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They are obtained from the Taylor expansion of

$$\tau_{n,j} = e^{\partial_j/n}, \quad \partial_{n,j} = n(e^{\partial_j/n} - 1) \quad \text{and} \quad \Delta_n = n^2 \sum_{j=1}^d \left(e^{\partial_j/n} + e^{-\partial_j/n} - 2 \right),$$

Modified heat equation

Recall that $\rho_t = \mathbb{E}\eta_t$ solves the PDE

$$\partial_t \rho(x) = \frac{1}{2} \Delta_n \rho(x), \quad x \in \mathbb{T}_n^d.$$

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As lift to the function space we consider the smooth interpolation

$$\text{ex}_n \rho_t := \sum_{k \in \mathbb{Z}_n^d} \langle \rho_t, \varsigma_k \rangle_n \varsigma_k$$

for the Fourier basis ς_k and $\mathbb{Z}_n^d = \{-N, \dots, N\}^d$, $n = 2N + 1$.

Modified heat equation

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for the Fourier basis ς_k and $\mathbb{Z}_n^d = \{-N, \dots, N\}^d$, $n = 2N + 1$.

We consider a solution ϱ_t to

$$\partial_t \varrho_t = \frac{1}{2} \Delta_n^{(2m)} \varrho_t$$

started from $\text{ex}_n \rho^0$.

Both solutions are started from the same function.

Approximation of particle expectations

Lemma

Let m be odd. Then for each n odd

$$\|\mathrm{ex}_n \rho_t - \varrho_t\|_{H_I} \leq \frac{Ct}{n^{2m}} \|\mathrm{ex}_n \rho_0\|_{H_{I+2m+2}}, \quad t \geq 0.$$

In particular,

$$\|\mathrm{ex}_n \rho_t - \varrho_t\|_{C^k(\mathbb{T}^d)} \leq \frac{C_k t}{n^{2m}} \|\mathrm{ex}_n \rho_0\|_{H_{k+\frac{d}{2}+2m+3}}, \quad t \geq 0.$$

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Proof follows from

$$\mathrm{ex}_n \rho_t^n = \sum_{k \in \mathbb{Z}_n^d} e^{\frac{1}{2} \lambda_{k,n} t} \langle \rho_0^n, s_k \rangle_n s_k, \quad \varrho_t^n = \sum_{k \in \mathbb{Z}_n^d} e^{\frac{1}{2} \lambda_{k,n}^{(2m)} t} \langle \rho_0^n, s_k \rangle_n s_k,$$

where

$$\lambda_{k,n} := n^2 \sum_{j=1}^d \left[\cos \frac{2\pi}{n} k_j - 1 \right], \quad \lambda_{k,n}^{(2m)} = \sum_{l=1}^m \frac{(-1)^l (2\pi)^{2l}}{(2l)! n^{2l-2}} \sum_{j=1}^d k_j^{2l} \leq 0.$$

Control of staying ϱ_t outside $[0, 1]$

Problem: $\text{ex}_n \rho^0(x) \notin [0, 1]$ and $\Delta_n^{(2m)}$ does not satisfies the max property.

Control of staying ϱ_t outside $[0, 1]$

Problem: $\text{ex}_n \rho^0(x) \notin [0, 1]$ and $\Delta_n^{(2m)}$ does not satisfies the max property.

Since

$$\Delta_n \varsigma_k = -\lambda_{k,n} \varsigma_k$$

on $\mathbb{T}^d$ (not only $\mathbb{T}_n^d$),

$$\partial_t \text{ex}_n \rho_t = \frac{1}{2} \Delta_n \text{ex}_n \rho_t$$

on $\mathbb{T}^d$ with initial condition $\text{ex}_n \rho^0$. Thus,

$$\min_{x \in \mathbb{T}^d} \text{ex}_n \rho^0(x) \leq \text{ex}_n \rho_t(x) \leq \max_{x \in \mathbb{T}^d} \text{ex}_n \rho^0(x) \quad (1)$$

for all $x \in \mathbb{T}^d$ and $t \geq 0$.

Control of staying ϱ_t outside $[0, 1]$

Problem: $\text{ex}_n \rho^0(x) \notin [0, 1]$ and $\Delta_n^{(2m)}$ does not satisfies the max property.

Since

$$\Delta_n \varsigma_k = -\lambda_{k,n} \varsigma_k$$

on $\mathbb{T}^d$ (not only $\mathbb{T}_n^d$),

$$\partial_t \text{ex}_n \rho_t = \frac{1}{2} \Delta_n \text{ex}_n \rho_t$$

on $\mathbb{T}^d$ with initial condition $\text{ex}_n \rho^0$. Thus,

$$\min_{x \in \mathbb{T}^d} \text{ex}_n \rho^0(x) \leq \text{ex}_n \rho_t(x) \leq \max_{x \in \mathbb{T}^d} \text{ex}_n \rho^0(x) \quad (1)$$

for all $x \in \mathbb{T}^d$ and $t \geq 0$.

If

$$\|\rho^0 - \text{ex}_n \rho^0\|_{C(\mathbb{T}^d)} \leq \frac{C}{n^{2m}} \|\rho^0\|_{C^{2m+1}(\mathbb{T}^d)}.$$

then

$$\begin{aligned} \varrho_t(x) &\leq \text{ex}_n \rho_t(x) + \|\varrho_t - \text{ex}_n \rho_t\|_{C(\mathbb{T}^d)} \\ &\leq \max_{y \in \mathbb{T}^d} \rho^0(y) + \frac{C(1+t)}{n^{2m}} \|\rho^0\|_{C^{d+2m+3}(\mathbb{T}^d)}. \end{aligned}$$

Approximation for ex_n

Lemma

For $f \in C^m(\mathbb{T}^d)$

$$\|f - \text{ex}_n f\|_{C(\mathbb{T}^d)} \leq \frac{C_m (1 + \ln^d n)}{n^m} \|f\|_{C^m(\mathbb{T}^d)}$$

and

$$|\langle f, s_k \rangle - \langle f, s_k \rangle_n| \leq \frac{C_m}{n^m} \|f\|_{C^m(\mathbb{T}^d)}$$

for all $k \in \mathbb{Z}_n^d$.

Idea of proof

Set

$$V_n = \text{span} \{ \varsigma_k, k \in \mathbb{Z}_n^d \} \subset C(\mathbb{T}^d).$$

Then

$$\text{ex}_n f(x) = \frac{1}{n^d} \sum_{y \in \mathbb{T}_n^d} \sum_{k \in \mathbb{Z}_n^d} f(y) \bar{\varsigma}_k(y) \varsigma_k(x) = \sum_{y \in \mathbb{T}_n^d} \phi(x, y) f(y),$$

where

$$\phi(x, y) = \frac{1}{n^d} \sum_{k \in \mathbb{Z}_n^d} \varsigma_k(x - y).$$

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$$\phi(x, y) = \frac{1}{n^d} \sum_{k \in \mathbb{Z}_n^d} \varsigma_k(x - y).$$

Using $\text{ex}_n g = g$ for $g \in V_n$, the estimate for Lebesgue constant and Jackson's theorem, we get

$$\begin{aligned} \|f - \text{ex}_n f\|_{C(\mathbb{T}^d)} &\leq \|f - g\|_{C(\mathbb{T}^d)} + \|g - \text{ex}_n f\|_{C(\mathbb{T}^d)} \\ &\leq (1 + \|\text{ex}_n\|_{op}) \|f - g\|_{C(\mathbb{T}^d)} \\ &\leq C (1 + \ln^d n) \frac{1}{n^m} \|f\|_{C^m(\mathbb{T}^d)} \end{aligned}$$

Diffusion term for OU process

Using $\eta = \rho + \frac{1}{n^{d/2}}\zeta$, we compute

$$\begin{aligned}(\tau_{n,j}\eta - \eta)^2 &= \underbrace{\eta + \tau_{n,j}\eta - 2\eta\tau_{n,j}\eta}_{=:\Gamma_{n,j}(\eta)} \\ &= \Gamma_{n,j}(\rho) + \frac{1}{n^{d/2}}\zeta + \frac{1}{n^{d/2}}\zeta\rho + \dots - \frac{1}{n^d}\zeta\tau_{n,j}\zeta\end{aligned}$$

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We introduce the modified OU process

$$\partial_t \xi_t = \frac{1}{2} \Delta_n^{(2m_\Delta)} \xi_t + \frac{1}{\sqrt{2}} \nabla_n^{(m_\partial)} \left(\sqrt{\Gamma_n^{(m_\tau)}(\varrho_t)^+} \dot{W}_t \right),$$

where

$$\Gamma_n^{(m_\tau)}(\varrho) = \varrho + \tau_{n,j}^{(m_\tau)} \varrho - 2\varrho \tau_{n,j}^{(m_\tau)} \varrho$$

New comparison of generators

$$\mathcal{L}_\rho^n F(\bar{\zeta}) = \frac{1}{2} \langle \Delta_n D F(\bar{\zeta}), \bar{\zeta} \rangle + \frac{1}{4} \sum_{j=1}^d \langle \partial_{n,j}^{\odot 2} D^2 F(\bar{\zeta}), (\tau_{n,j} \eta - \eta)^2 \rangle_n + O\left(\frac{1}{n^{d/2+1}}\right)$$

$$\mathcal{G}_\varrho^n F(\xi) = \frac{1}{2} \langle \Delta_n^{(2m_\Delta)} D F(\xi), \xi \rangle + \frac{1}{4} \sum_{j=1}^d \left\langle \left(\partial_{n,j}^{(m_\partial)} \right)^{\odot 2} D^2 F(\xi), \Gamma_{n,j}^{(m_\tau)}(\varrho)^+ \right\rangle$$

$$\langle \varphi, \Gamma_{n,j}(\eta) \rangle_n - \left\langle \varphi, \Gamma_{n,j}^{(m_\tau)}(\varrho)^+ \right\rangle$$

$$\frac{1}{n^d} \zeta \tau_{n,j} \zeta + [\text{fluctuation err.}] = \langle \varphi, \Gamma_{n,j}(\eta) - \Gamma_{n,j}(\rho) \rangle_n$$

$$[\text{profile err.}] + \langle \varphi, \Gamma_{n,j}(\rho) - \Gamma_{n,j}(\varrho) \rangle_n$$

$$[\text{noise discretisation err.}] + [\langle \varphi, \Gamma_{n,j}(\varrho) \rangle_n - \langle \varphi, \Gamma_{n,j}(\varrho) \rangle]$$

$$[\text{positive correction}] + \langle \varphi, \Gamma_{n,j}(\varrho) - \Gamma_{n,j}(\varrho)^+ \rangle$$

$$[\text{shift approx. err.}] + \left\langle \varphi, \Gamma_{n,j}(\varrho)^+ - \Gamma_{n,j}^{(m_\tau)}(\varrho)^+ \right\rangle$$

All Errors

$$\mathcal{C}^n F(\bar{\zeta}) = \frac{1}{2n^d} \left\langle \left(\nabla_n^{(m_\partial)} \right)^{\odot 2} D^2 F(\bar{\zeta}), \zeta \tau_n \zeta \right\rangle_n$$

Theorem [Gess, K. '26+]

Let $I > \frac{d}{2}$ $F \in C^3(H_{-I})$ with kernels $D^k F(g; \cdot) \in C^m(\mathbb{T}^{dk})$. Then

$$\begin{aligned} |(\mathcal{L}_\rho^n - \mathcal{G}_\varrho^n + \mathcal{C}^n) F(\bar{\zeta})| &\leq C \left(E_\Delta^n + E_\partial^n + E_{\text{fluc}}^n + E_{\text{lln}}^n + E_{\text{noise}}^n \right. \\ &\quad \left. + E_+^n + E_\tau^n + R^n \right) \|F(\bar{\zeta})\|_{C^{3,m}}, \end{aligned}$$

The error terms are given by

$$\begin{aligned} E_\Delta^n &:= \frac{1}{n^{2m_\Delta}} \|\bar{\zeta}\|_{H_{-I}}, & E_\partial^n &:= \frac{1}{n^{m_\partial}} \\ E_{\text{fluc}}^n &:= \frac{1}{n^{\frac{d}{2}}} \|\bar{\zeta}\|_{H_{-I}}, & E_{\text{lln}}^n &:= \|\text{ex}_n \rho - \varrho\|_{C(\mathbb{T}^d)} \\ E_{\text{noise}}^n &:= \frac{1}{n^{m_W}} \|\varrho\|_{C^m(\mathbb{T}^d)}^2, & E_\tau^n &:= \frac{1}{n^{m_\tau+1}} \|\varrho\|_{C^m(\mathbb{T}^d)} \\ E_+^n &:= \left(\|\varrho^-\|_{C(\mathbb{T}^d)} + \|(1-\varrho)^-\|_{C(\mathbb{T}^d)} \right), & R^n &:= \frac{1}{n^{\frac{d}{2}+1}}. \end{aligned}$$

Duhamel's principle

Let $\xi_{s,\cdot}(g)$ be a solution to the modified OU equation started from g at time s .
Set

$$\mathcal{P}_{s,t}F(g) = \mathbb{E}F(\xi_{s,t}(g)), \quad g \in H_{-I}.$$

Duhamel's principle

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Set

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Using the martingale problem for ζ_t and the backward Kolmogorov equation, we obtain

$$\begin{aligned}\mathbb{E}F(\bar{\zeta}_t^n) - \mathbb{E}F(\xi_t^n) &= \mathbb{E}\mathcal{P}_{t,t}F(\bar{\zeta}_t^n) - \mathbb{E}\mathcal{P}_{0,t}F(\xi_0^n) \\&= \mathbb{E}\mathcal{P}_{0,t}F(\bar{\zeta}_0^n) - \mathbb{E}\mathcal{P}_{0,t}F(\xi_0^n) + \mathbb{E} \int_0^t (\partial_s + \mathcal{L}_{\rho_s}^n) \mathcal{P}_{s,t}^n F(\bar{\zeta}_s^n) ds \\&= \mathbb{E}\mathcal{P}_{0,t}F(\bar{\zeta}_0^n) - \mathbb{E}\mathcal{P}_{0,t}F(\xi_0^n) + \mathbb{E} \int_0^t (\mathcal{L}_{\rho_s}^n - \mathcal{G}_{\varrho_s}^n) \mathcal{P}_{s,t}^n F(\bar{\zeta}_s^n) ds \\&= \mathbb{E}\mathcal{P}_{0,t}F(\bar{\zeta}_0^n) - \mathbb{E}\mathcal{P}_{0,t}F(\xi_0^n) + \mathbb{E} \int_0^t (\mathcal{L}_{\rho_s}^n - \mathcal{G}_{\varrho_s}^n + \mathcal{C}^n) \mathcal{P}_{s,t}^n F(\bar{\zeta}_s^n) ds \\&\quad - \mathbb{E} \int_0^t \mathcal{C}^n \mathcal{P}_{s,t}^n F(\bar{\zeta}_s^n) ds.\end{aligned}$$

Main assumptions

- $m_\partial, m_W, m_\rho \in \mathbb{N}, m_\tau \in \mathbb{N}_0, m_\Delta \in 2\mathbb{N} - 1$ and $I > \frac{d}{2} + m_\partial$
- $M_\rho, M_k, k \in [3]$, be large enough
- $\rho^0 \in C^{M_\rho}(\mathbb{T}^d)$ and $F \in C^3(H_{-I})$ such that

$$K_F := \max_{k \in [3]} \sup_{g \in H_{-I}} [\|D^k F(g)\|_{C^{M_k}}] < \infty$$

- Let η_t^n be a SSEP started from local equilibrium with profile ρ^0 and

$$\bar{\zeta}_t^n = \frac{1}{n^d} \sum_{x \in \mathbb{T}_n^d} (\eta_t^n(x) - \mathbb{E} \eta_t^n(x)) \delta_x$$

- ϱ_t solves the PDE

$$\partial_t \varrho_t = \frac{1}{2} \Delta_n^{(2m_\Delta)} \varrho_t, \quad \varrho_0 = \rho^0$$

- ξ_t^n be a solution in H_{-I} to

$$\partial_t \xi_t = \frac{1}{2} \Delta_n^{(2m_\Delta)} \xi_t + \frac{1}{\sqrt{2}} \nabla_n^{(m_\partial)} \left(\sqrt{\Gamma_n^{(m_\tau)}} (\varrho_t)^+ \dot{W}_t \right),$$

started from a centered Gaussian random variable in H_{-I} with covariance

$$\mathbb{E} [\langle \xi, \varphi \rangle \langle \xi, \psi \rangle] = \langle \rho^0 (1 - \rho^0) \varphi, \psi \rangle$$

Main result

Recall

$$\mathcal{C}^n F(\bar{\zeta}) = \frac{1}{2n^d} \left\langle (\nabla_n^{(m_\partial)})^{\odot 2} D^2 F(\bar{\zeta}), \zeta \tau_n \zeta \right\rangle_n$$

and

$$\mathcal{P}_{s,t} F(g) = \mathbb{E} F(\xi_{s,t}(g))$$

Theorem [Gess, K. '26+]

Under the assumptions above, one has

$$\begin{aligned} & \left| \mathbb{E} F(\bar{\zeta}_t^n) - \mathbb{E} F(\xi_t^n) + \mathbb{E} \int_0^t \mathcal{C}^n \mathcal{P}_{s,t}^n F(\bar{\zeta}_s^n) ds \right| \\ & \leq \frac{C_{d,I}}{n^{d/2}} K_F + \frac{C(1+t)}{n^\alpha} t K_F \left(1 + \|\rho^0\|_{C^{M_\rho}(\mathbb{T}^d)} \right), \end{aligned}$$

where

$$\alpha := \min \left\{ \frac{d}{2}, 2m_\Delta, m_\partial, m_W, m_\tau + 1 \right\}.$$

Control of correlation correction term

It is known

$$\sup_{x_i \text{ dist.}} \mathbb{E} \left[\prod_{i=1}^k (\eta_t(x_i) - \rho_t(x_i)) \right] \leq C \begin{cases} \frac{1}{n^{k/2}}, & d = 1 \\ \frac{1}{n^k} \ln^{k-1}(1 + n^2 t), & d = 2 \\ \frac{1}{n^k}, & d \geq 3 \end{cases}$$

see [Ferrari, Presutti, Scacciatelli, Vares '91] for $d = 1$ and [Xu, Zhao '25] for $d \geq 2$.

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see [Ferrari, Presutti, Scacciatelli, Vares '91] for $d = 1$ and [Xu, Zhao '25] for $d \geq 2$.

Thus,

$$\left| \mathbb{E} \int_0^t C^n \mathcal{P}_{s,t}^n F(\bar{\zeta}_s^n) ds \right| \leq \frac{C}{n^{\frac{d}{2} \wedge 2}} K_F$$

References

- [1] Benjamin Gess and Vitalii Konarovskiy.
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to appear in Ann. Prob.
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Thank you!