

Higher-Order Gaussian Approximations on Path Space for Simple Symmetric Exclusion Process

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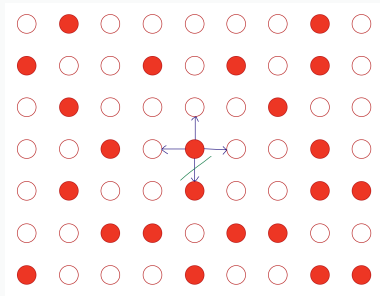
DER FORSCHUNG | DER LEHRE | DER BILDUNG

Simple symmetric exclusion process

On the d -dim discrete torus

$$\mathbb{T}_n^d := \left\{ \frac{k}{n} : k \in \{0, \dots, n-1\}^d \right\} \subset \mathbb{T}^d = (\mathbb{R}/\mathbb{Z})^d$$

we consider a **Simple Symmetric Exclusion Process (SSEP)**

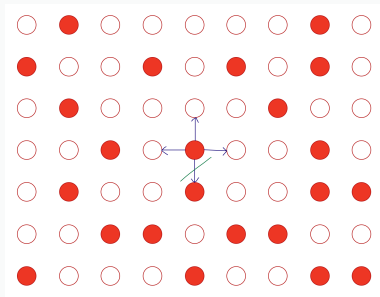


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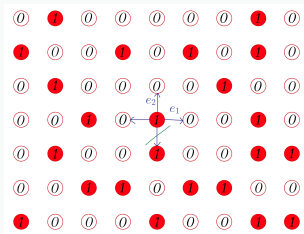


Particles make jumps to nearest-neighbour unoccupied slots at exponential times.

Diffusive scaling and generator

A configuration is $\eta \in \{0, 1\}^{\mathbb{T}_n^d}$,

$$\eta(x) = 0 \Leftrightarrow x \text{ is empty}, \quad \eta(x) = 1 \Leftrightarrow x \text{ is occupied}$$

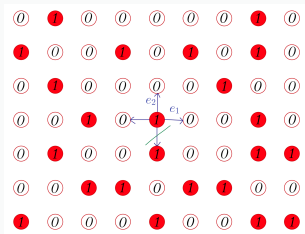


$$\eta^{x \leftrightarrow y}(z) = \begin{cases} \eta(z), & z \neq x, y, \\ \eta(y), & z = x, \\ \eta(x), & z = y, \end{cases}$$

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$$\mathcal{L}^n F(\eta) := \frac{n^2}{2} \sum_{j=1}^d \sum_{x \in \mathbb{T}_n} [F(\eta^{x \leftrightarrow x+e_j}) - F(\eta)] \quad [\text{Kipnis, Landim '99}]$$

Space is scaled by $\frac{1}{n}$ and time by n^2

Hydrodynamic limit

Let $\varrho_* : \mathbb{T}^d \rightarrow [0, 1]$ and assume

$$\eta_0^n(x) \sim \text{Bernulli}(\varrho_*(x)), \quad x \in \mathbb{T}_n^d,$$

independently.

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The expectation

$$\rho_t^n(x) := \mathbb{E} \eta_t^n(x)$$

solves the discrete heat equation

$$\partial_t \rho_t^n = \frac{1}{2} \Delta_n \rho_t^n.$$

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For the empirical density

$$\bar{\eta}_t^n := \frac{1}{n^d} \sum_{x \in \mathbb{T}_n^d} \eta_t^n(x) \delta_x,$$

we have

$$\bar{\eta}_t^n \longrightarrow \varrho_t(x) dx, \quad \text{where} \quad \partial_t \varrho_t = \frac{1}{2} \Delta \varrho_t, \quad \varrho_0 = \varrho_*.$$

[see e.g. Kipnis–Landim '99]

Fluctuations and classical CLT

We center η^n around the deterministic mean and use the CLT scaling:

$$\zeta_t^n(x) := n^{d/2}(\eta_t^n(x) - \rho_t^n(x)), \quad \bar{\zeta}_t^n := \frac{1}{n^d} \sum_{x \in \mathbb{T}_n^d} \zeta_t^n(x) \delta_x.$$

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Theorem [Galves–Kipnis–Spohn; Ravishankar '90]

If ϱ_* is smooth, then

$$(\bar{\zeta}_t^n)_{t \in [0, T]} \Longrightarrow (\xi_t)_{t \in [0, T]} \quad \text{in } D([0, T], \mathcal{D}'),$$

where ξ solves

$$\partial_t \xi_t = \frac{1}{2} \Delta \xi_t + \operatorname{div} \left(\sqrt{\varrho_t(1 - \varrho_t)} \dot{W}_t \right)$$

with the centered Gaussian initial condition satisfying

$$\mathbb{E} [\langle \xi_*, \varphi \rangle^2] = \langle \varrho_*(1 - \varrho_*) \varphi, \varphi \rangle$$

Quantitative CLT at fixed time

Theorem [Gess–K. '26, AOP]

For smooth enough ϱ_* , f and φ ,

$$\sup_{t \in [0, T]} \left| \mathbb{E} f(\langle \varphi, \bar{\xi}_t^n \rangle) - \mathbb{E} f(\langle \varphi, \xi_t \rangle) \right| \leq \frac{C}{n^{\frac{d}{2} \wedge 1}} \|f\|_{C_b^3} \|\varphi\|_{C^1}.$$

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There are two different errors:

$$\underbrace{n^{-d/2}}_{\text{particle / CLT error}} \quad \text{and} \quad \underbrace{n^{-1}}_{\text{lattice discretization error}}.$$

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Can we obtain a path-space estimate and,

at the same time, go beyond n^{-1} ?

Particle generator vs. OU generator

For $\zeta = n^{d/2}(\eta - \rho)$ and $F \in C^3(H^{-I})$

Particle generator:

$$\begin{aligned}\mathcal{L}_\rho^n F(\bar{\zeta}) &= \frac{1}{2} \langle \Delta_n D F(\bar{\zeta}), \bar{\zeta} \rangle + \frac{1}{4} \sum_{j=1}^d \langle \partial_{n,j}^{\odot 2} D^2 F(\bar{\zeta}; \cdot, \cdot), \eta + \tau_{n,j} \eta - 2\eta \tau_{n,j} \eta \rangle_n \\ &\quad + O\left(\frac{1}{n^{d/2+1}}\right)\end{aligned}$$

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Generator associated with

$$\partial_t \xi_t = \frac{1}{2} \Delta \xi_t + \operatorname{div} \left(\sqrt{\varrho_t(1 - \varrho_t)} \dot{W}_t \right)$$

is given by

$$\mathcal{G}_\varrho F(\bar{\zeta}) = \frac{1}{2} \langle \Delta D F(\bar{\zeta}), \bar{\zeta} \rangle + \frac{1}{2} \sum_{j=1}^d \langle \partial_j^{\odot 2} D^2 F(\bar{\zeta}), \varrho_t(1 - \varrho_t) \rangle$$

Removing the discretization barrier

The lattice operators can be written formally as

$$\tau_{n,j} = e^{\partial_j/n}, \quad \partial_{n,j} = n(e^{\partial_j/n} - 1),$$

$$\Delta_n = n^2 \sum_{j=1}^d (e^{\partial_j/n} + e^{-\partial_j/n} - 2).$$

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Instead of keeping only the first continuum term, approximate

$$e^z \text{ by its Taylor polynomial } P_m(z)$$

This gives higher-order differential operators

$$\tau_{n,j}^{(m_\tau)} = P_{m_\tau}(\partial_j/n), \quad \partial_{n,j}^{(m_\partial)} = \dots, \quad \Delta_n^{(2m_\Delta)} = \dots$$

with errors of orders

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The Gaussian approximation now retains higher-order lattice information.

Modified Gaussian approximation

First replace the macroscopic profile by

$$\partial_t \varrho_t^n = \frac{1}{2} \Delta_n^{(2m_\Delta)} \varrho_t^n, \quad \varrho_0^n = \varrho_*.$$

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We then approximate $\bar{\zeta}^n$ by the Gaussian process ξ^n solving

$$\partial_t \xi_t^n = \frac{1}{2} \Delta_n^{(2m_\Delta)} \xi_t^n + \frac{1}{\sqrt{2}} \operatorname{div}_n^{(m_\partial)} \left(\sqrt{\Gamma_n^{(m_\tau)}(\varrho_t^n)^+} \dot{W}_t \right).$$

Here, componentwise,

$$\Gamma_{n,j}^{(m_\tau)}(\varrho) = \varrho + \tau_{n,j}^{(m_\tau)} \varrho - 2\varrho \tau_{n,j}^{(m_\tau)} \varrho.$$

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For $n \rightarrow \infty$, ξ^n still converges to the classical OU process, but it is a much more accurate approximation at finite n .

Path-space observables

We test the entire trajectory by **three times Fréchet differentiable** functions

$$F : D([0, T], H^{-I}(\mathbb{T}^d)) \rightarrow \mathbb{R}$$

such that

$$D^k F(f) [\mathbb{I}_{[t_n, t)} \cdot, \cdot, \dots, \cdot] \rightarrow 0, \quad t_n \uparrow t,$$

where $D([0, T], H^{-I}(\mathbb{T}^d))$ is equipped with uniform topology.

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Two basic examples are

$$F(\gamma) = G(\gamma_{t_1}, \dots, \gamma_{t_N}), \quad 0 \leq t_1 < \dots < t_N \leq T,$$

and

$$F(\gamma) = G\left(\int_{[0, T]} \gamma_s \nu(ds)\right).$$

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We work with a class $\mathcal{P}^3(D([0, T], H^{-I}(\mathbb{T}^d)))$ of such functions F .

Quantitative functional CLT

Theorem [Gess–K. '26+]

Let

$$\mathcal{P}^3 \left(D([0, T], H^{-I}(\mathbb{T}^d)) \right)$$

and let ϱ_* be smooth enough. Then

$$\begin{aligned} & \left| \mathbb{E} F \left((\tilde{\zeta}_s^n)_{s \leq T} \right) - \mathbb{E} F \left((\xi_s^n)_{s \leq T} \right) \right| \\ & \leq C \left(\frac{1}{n^{\frac{d}{2}}} + \frac{1}{n^\alpha} + \frac{1}{n^{\frac{d}{2} \wedge 2}} \right) \|F\|_{C^3(D([0, T], H^{-I}))}, \end{aligned}$$

where

$$\alpha = \min \{ 2m_\Delta, m_\partial, m_\tau + 1 \}.$$

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Here are three different errors:

$$\underbrace{n^{-d/2}}_{\text{particle / CLT error}},$$

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$$\text{and } \underbrace{n^{-\frac{d}{2} \wedge 2}}_{\text{correlation error}}.$$

Continuation functional on path space

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For a path functional F , define a non-anticipative functional

$$U_t(\gamma) := \mathbb{E} F \left(\gamma^{[0,t)} + (\xi_{t,\cdot}^n(\gamma_t))^{[t,T]} \right)$$

for $t \in [0, T]$, $\gamma \in D([0, t], H^{-I})$.

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Interpretation:

- keep the past $\gamma^{[0,t)} = \gamma|_{[0,t)}$ fixed,
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In particular,

$$U_T(\gamma) = F(\gamma).$$

This is the path-space analogue of the backward semigroup.

Horizontal regularity on path space

For a stopped path $\gamma \in D([0, t], E)$, let

$$(\Theta_{t,r}\gamma)_s = \gamma_{s \wedge r}, \quad 0 \leq s \leq r$$

denote its constant extension from t to r .

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For path functionals, the classical **horizontal derivative**

$$\mathcal{D}_t U(\gamma) = \lim_{r \rightarrow t+} \frac{U_r(\Theta_{t,r}\gamma) - U_t(\gamma)}{r - t}$$

may fail to exist at prescribed times.

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We therefore use **horizontal absolute continuity**:

$$G_r(\Theta_{t,r}\gamma) - G_t(\gamma) = \int_t^r \mathcal{H}_s G(\Theta_{t,s}\gamma) ds.$$

Backward Kolmogorov equation on path space

For the continuation function

$$U_t(\gamma) = \mathbb{E} F \left(\gamma^{[0,t)} + \xi_{t,\cdot}^{n,[t,T]}(\gamma_t) \right),$$

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Proposition [Gess–K. '26+]

The continuation functional U is horizontally absolutely continuous and

$$\mathcal{H}_t U + \mathcal{G}_{\varrho_t} U_t = 0,$$

Equivalently,

$$U_r(\gamma) = U_t(\Theta_{r,t}\gamma) + \int_r^t \mathcal{G}_{\varrho_s} U_s(\Theta_{r,s}\gamma) ds.$$

Path-dependent martingale problem

Let Y be a Markov jump process with generator $\mathcal{L}$ and let G be a left-continuous horizontally absolutely continuous non-anticipative functional.

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Proposition [Gess–K. '26+]

Under a mild integrability conditions,

$$G_t(Y^{[0,t]}) - G_0(Y_0) - \int_0^t \left[\mathcal{H}_s G(Y^{[0,s]}) + \mathcal{L}G_s(Y^{[0,s]}, \cdot)(Y_s) \right] ds$$

is a local martingale.

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This is the path-dependent analogue of Dynkin's formula.

Path-dependent Trotter formula

Apply the martingale problem to the SSEP and the continuation functional U , and use

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After taking expectations,

$$\begin{aligned} & \mathbb{E}F(\bar{\zeta}^{n,[0,T]}) - \mathbb{E}F(\xi^{n,[0,T]}) \\ &= \mathbb{E}\left[U_0^n(\bar{\zeta}_0^n) - U_0^n(\xi_*)\right] + \mathbb{E}\int_0^T (\mathcal{L}_{\rho_s}^n + \mathcal{H}_s^n) U_s^n(\bar{\zeta}^{n,[0,s]}) ds \\ &= \underbrace{\mathbb{E}\left[U_0^n(\bar{\zeta}_0^n) - U_0^n(\xi_*)\right]}_{\text{initial Gaussian approximation}} + \underbrace{\mathbb{E}\int_0^T (\mathcal{L}_{\rho_s}^n - \mathcal{G}_{\varrho_s}^n) U_s^n(\bar{\zeta}^{n,[0,s]}) ds}_{\text{generator mismatch}}. \end{aligned}$$

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This is a Trotter-type formula on path space.

The same mechanism is potentially useful far beyond SSEP.

Generator comparison

The microscopic generator can be matched with the modified Gaussian one:

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The leading non-Gaussian term is

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All remaining terms are controlled by

$$\frac{1}{n^{d/2}} + \frac{1}{n^{2m_\Delta}} + \frac{1}{n^{m_\partial}} + \frac{1}{n^{m_\tau+1}}.$$

Higher-order differential operator matching removes the old n^{-1} discretization error.

Exclusion correlations

The correction $\mathcal{C}$ contains nearest-neighbour products

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One proves

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Hence, after arbitrarily accurate lattice matching, the remaining error is

$$n^{-(\frac{d}{2} \wedge 2)}.$$

Moreover, for non-constant profiles the nearest-neighbour covariance can be of order

$$|\text{Cov}(\eta_t(x), \eta_t(x + e_j/n))| \gtrsim n^{-2},$$

so the n^{-2} barrier in high dimension is genuine.

Conclusions

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3. General path-space method:

backward Kolmogorov equation + path-dependent martingale problem

$\implies$ Trotter-type formula

This reduces a path-space weak approximation problem to an initial Gaussian approximation and a generator comparison.

References

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Thank you!