

Higher-Order Gaussian Approximations on Path Space for the Simple Symmetric Exclusion Process

Vitalii Konarovskiy

Joint work with: Benjamin Gess (TU Berlin & MPI Leipzig)

Universität Hamburg

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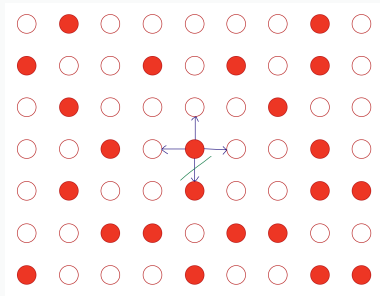
DER FORSCHUNG | DER LEHRE | DER BILDUNG

Simple symmetric exclusion process

On the d -dim discrete torus

$$\mathbb{T}_n^d := \left\{ \frac{k}{n} : k \in \{0, \dots, n-1\}^d \right\} \subset \mathbb{T}^d = (\mathbb{R}/\mathbb{Z})^d$$

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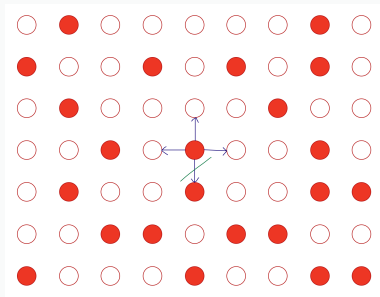


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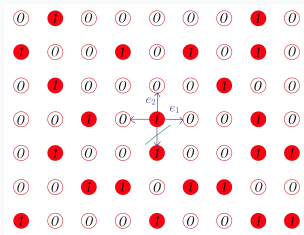


Particles make jumps to nearest-neighbour unoccupied slots at exponential times.

Diffusive scaling and generator

A configuration is $\eta \in \{0, 1\}^{\mathbb{T}_n^d}$,

$$\eta(x) = 0 \Leftrightarrow x \text{ is empty}, \quad \eta(x) = 1 \Leftrightarrow x \text{ is occupied}$$

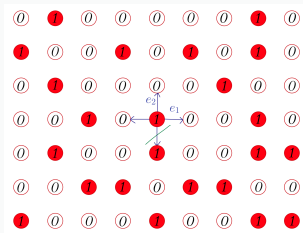


$$\eta^{x \leftrightarrow y}(z) = \begin{cases} \eta(z), & z \neq x, y, \\ \eta(y), & z = x, \\ \eta(x), & z = y, \end{cases}$$

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$$\mathcal{L}^n F(\eta) := \frac{n^2}{2} \sum_{j=1}^d \sum_{x \in \mathbb{T}_n^d} [F(\eta^{x \leftrightarrow x+e_j/n}) - F(\eta)] \quad [\text{Kipnis, Landim '99}]$$

Space is scaled by $\frac{1}{n}$ and time by n^2

Hydrodynamic limit

Let $\varrho_* : \mathbb{T}^d \rightarrow [0, 1]$ and assume

$$\eta_0^n(x) \sim \text{Bernoulli}(\varrho_*(x)), \quad x \in \mathbb{T}_n^d,$$

independently.

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independently.

The expectation

$$\rho_t^n(x) := \mathbb{E} \eta_t^n(x)$$

solves the discrete heat equation

$$\partial_t \rho_t^n = \frac{1}{2} \Delta_n \rho_t^n.$$

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For the empirical density

$$\bar{\eta}_t^n := \frac{1}{n^d} \sum_{x \in \mathbb{T}_n^d} \eta_t^n(x) \delta_x,$$

we have

$$\bar{\eta}_t^n \longrightarrow \varrho_t(x) dx, \quad \text{where} \quad \partial_t \varrho_t = \frac{1}{2} \Delta \varrho_t, \quad \varrho_0 = \varrho_*.$$

[see e.g. Kipnis–Landim '99]

Fluctuations and classical CLT

Centering by the exact lattice mean ρ_t^n separates fluctuations from the deterministic spatial error:

$$\zeta_t^n(x) := n^{d/2}(\eta_t^n(x) - \rho_t^n(x)), \quad \bar{\zeta}_t^n := \frac{1}{n^d} \sum_{x \in \mathbb{T}_n^d} \zeta_t^n(x) \delta_x.$$

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Theorem [Galves–Kipnis–Spohn; Ravishankar '92]

If ϱ_* is smooth, then

$$(\bar{\zeta}_t^n)_{t \in [0, T]} \Longrightarrow (\xi_t)_{t \in [0, T]} \quad \text{in } D([0, T], \mathcal{D}'),$$

where ξ solves

$$\partial_t \xi_t = \frac{1}{2} \Delta \xi_t + \operatorname{div} \left(\sqrt{\varrho_t(1 - \varrho_t)} \dot{W}_t \right)$$

with the centered Gaussian initial condition satisfying

$$\mathbb{E} [\langle \xi_*, \varphi \rangle^2] = \langle \varrho_*(1 - \varrho_*) \varphi, \varphi \rangle$$

Quantitative CLT at fixed time

Theorem [Gess–K. '26, AOP]

For sufficiently smooth ϱ_* , f and φ ,

$$\sup_{t \in [0, T]} \left| \mathbb{E} f(\langle \varphi, \bar{\zeta}_t^n \rangle) - \mathbb{E} f(\langle \varphi, \xi_t \rangle) \right| \leq C_{T, \varrho_*, f, \varphi} n^{-\min\{d/2, 1\}}.$$

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The estimate separates two contributions:

$$\underbrace{n^{-d/2}}_{\text{particle / CLT error}} \quad \text{and} \quad \underbrace{n^{-1}}_{\text{lattice approximation error}}.$$

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Can an n -dependent Gaussian process approximate the entire fluctuation trajectory beyond order n^{-1} ?

Particle generator vs. OU generator

For $\zeta = n^{d/2}(\eta - \rho)$ and $F \in C^3(H^{-I})$

Particle generator:

$$\begin{aligned}\mathcal{L}_\rho^n F(\bar{\zeta}) &= \frac{1}{2} \langle \Delta_n D F(\bar{\zeta}), \bar{\zeta} \rangle + \frac{1}{4} \sum_{j=1}^d \langle \partial_{n,j}^{\odot 2} D^2 F(\bar{\zeta}; \cdot, \cdot), \eta + \tau_{n,j} \eta - 2\eta \tau_{n,j} \eta \rangle_n \\ &\quad + O\left(\frac{1}{n^{d/2+1}}\right)\end{aligned}$$

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Generator associated with

$$\partial_t \xi_t = \frac{1}{2} \Delta \xi_t + \operatorname{div} \left(\sqrt{\varrho_t(1 - \varrho_t)} \dot{W}_t \right)$$

is given by

$$\mathcal{G}_\varrho F(\bar{\zeta}) = \frac{1}{2} \langle \Delta D F(\bar{\zeta}), \bar{\zeta} \rangle + \frac{1}{2} \sum_{j=1}^d \langle \partial_j^{\odot 2} D^2 F(\bar{\zeta}), \varrho_t(1 - \varrho_t) \rangle$$

Removing the discretization barrier

The lattice operators can be written formally as

$$\tau_{n,j} = e^{\partial_j/n}, \quad \partial_{n,j} = n(e^{\partial_j/n} - 1),$$

$$\Delta_n = n^2 \sum_{j=1}^d (e^{\partial_j/n} + e^{-\partial_j/n} - 2).$$

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Replace exponentials by Taylor polynomials P_m :

$$\tau_{n,j}^{(m_\tau)} = P_{m_\tau}(\partial_j/n), \quad \partial_{n,j}^{(m_\partial)} = n(P_{m_\partial}(\partial_j/n) - 1).$$

For example, with $m_\Delta = 3$,

$$\Delta_n^{(6)} = \Delta + \frac{1}{12n^2} \sum_j \partial_j^4 + \frac{1}{360n^4} \sum_j \partial_j^6.$$

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Choose **odd** m_Δ so the modified Laplacian is dissipative. The shift, derivative and Laplacian errors have orders

$$n^{-(m_\tau+1)}, \quad n^{-m_\partial}, \quad n^{-2m_\Delta}.$$

Modified Gaussian approximation

The modified density solves

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$$\partial_t \xi_t^n = \frac{1}{2} \Delta_n^{(2m_\Delta)} \xi_t^n + \frac{1}{\sqrt{2}} \operatorname{div}_n^{(m_\partial)} \left(\sqrt{\Gamma_n^{(m_\tau)}(\varrho_t^n)} \dot{W}_t \right).$$

Here, componentwise,

$$\Gamma_{n,j}^{(m_\tau)}(\varrho) = \varrho + \tau_{n,j}^{(m_\tau)} \varrho - 2\varrho \tau_{n,j}^{(m_\tau)} \varrho.$$

The positive part ensures a nonnegative noise covariance.

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The positive part ensures a nonnegative noise covariance.

The coefficients are deterministic: ξ^n remains Gaussian.

As $n \rightarrow \infty$, it has the same classical OU limit, while retaining more lattice information at finite n .

Path-space observables

We compare smooth functionals of the entire trajectory:

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The class $\mathcal{P}^3$ has bounded, left-continuous path derivatives up to order three. We use the uniform norm on path space.

Quantitative functional CLT

Theorem [Gess–K. '26+]

Let m_Δ be odd, I sufficiently large, ϱ_* sufficiently smooth, and

$$F \in \mathcal{P}^3\left(D([0, T], H^{-I}(\mathbb{T}^d))\right).$$

Then

$$\begin{aligned} & \left| \mathbb{E}F(\bar{\zeta}^{n, [0, T]}) - \mathbb{E}F(\xi^{n, [0, T]}) \right| \\ & \leq C \left(n^{-d/2} + n^{-\beta} + n^{-\min\{d/2, 2\}} \right) \|F\|_{\mathcal{P}^3}, \end{aligned}$$

where $\beta = \min\{2m_\Delta, m_\partial, m_\tau + 1\}$.

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where $\beta = \min\{2m_\Delta, m_\partial, m_\tau + 1\}$.

There are three contributions:

$$\underbrace{n^{-d/2}}_{\text{CLT error}}, \quad \underbrace{n^{-\beta}}_{\text{spatial approximation}}, \quad \underbrace{n^{-\min\{d/2, 2\}}}_{\text{correlations}}.$$

If $\beta \geq d/2$, the direct weak error is $O(n^{-\min\{d/2, 2\}})$.

Related work

Cont–Fournié (2010)

Change of variable formulas for non-anticipative functionals on path space.

Horizontal and vertical derivatives; functional change-of-variable formulas.

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Weak approximation of independent diffusing particles by discretized or regularized nonlinear SPDEs.

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Weak approximation of independent diffusing particles by discretized or regularized nonlinear SPDEs.

Here: SSEP fluctuation trajectories versus modified Gaussian dynamics, with an explicit microscopic correlation correction.

Continuation functional on path space

For an observed history $\gamma \in D([0, t], H^{-I})$, define

$$U_t^n(\gamma) := \mathbb{E} F \left(\gamma^{[0, t)} + (\xi_{t, s}^n(\gamma_t))_{s \in [t, T]} \right).$$

Here $\xi_{t, s}^n(g)$ starts from g at time t .

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- Keep the observed past $\gamma^{[0, t)}$ fixed.
- Continue from γ_t with the modified Gaussian dynamics.
- Evaluate F and average over the future Gaussian noise.

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- Keep the observed past $\gamma^{[0, t)}$ fixed.
- Continue from γ_t with the modified Gaussian dynamics.
- Evaluate F and average over the future Gaussian noise.

In particular,

$$U_T^n(\gamma) = F(\gamma).$$

This is the path-space analogue of the backward semigroup.

Horizontal regularity on path space

For $\gamma \in D([0, t], E)$ and $r \geq t$, its constant extension is

$$(\Theta_{t,r}\gamma)_s = \gamma_{s \wedge t}, \quad 0 \leq s \leq r.$$

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$$(\Theta_{t,r}\gamma)_s = \gamma_{s \wedge t}, \quad 0 \leq s \leq r.$$

The classical **horizontal derivative** is

$$\mathcal{D}_t G(\gamma) = \lim_{r \downarrow t} \frac{G_r(\Theta_{t,r}\gamma) - G_t(\gamma)}{r - t},$$

when this limit exists.

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when this limit exists.

For the comparison, we use an integral backward equation. The relevant notion is **horizontal absolute continuity**:

$$G_r(\Theta_{t,r}\gamma) - G_t(\gamma) = \int_t^r \mathcal{H}_s G(\Theta_{t,s}\gamma) ds.$$

The horizontal derivative $\mathcal{H}_s G$ is needed only almost everywhere.

Backward Kolmogorov equation on path space

The continuation functional is

$$U_t^n(\gamma) = \mathbb{E}F\left(\gamma^{[0,t]} + (\xi_{t,s}^n(\gamma_t))_{s \in [t,T]}\right).$$

Backward Kolmogorov equation on path space

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Proposition [Gess–K. '26+]

It is horizontally absolutely continuous and satisfies

$$\mathcal{H}_t U^n + \mathcal{G}_{\varrho_t^n} U_t^n = 0$$

for almost every t , along constant extensions of admissible histories. Equivalently, for $r \leq t$,

$$U_r^n(\gamma) = U_t^n(\Theta_{r,t}\gamma) + \int_r^t \mathcal{G}_{\varrho_s^n} U_s^n(\Theta_{r,s}\gamma) ds.$$

The Gaussian generator acts on the terminal state, with the past fixed.

Path-dependent martingale problem

Let Y be a Markov jump process with generator $\mathcal{L}$ and let G be a left-continuous horizontally absolutely continuous non-anticipative functional.

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Proposition [Gess–K. '26+]

Under suitable integrability conditions,

$$G_t(Y^{[0,t]}) - G_0(Y_0) - \int_0^t \left[\mathcal{H}_s G(Y^{[0,s]}) + \mathcal{L}G_s(Y^{[0,s]}, \cdot)(Y_s) \right] ds$$

is a local martingale.

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is a local martingale.

This is the path-dependent analogue of Dynkin's formula.

Path-dependent Trotter formula

Combine the martingale formula along the microscopic trajectory with

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Path-dependent Trotter formula

Combine the martingale formula along the microscopic trajectory with

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After taking expectations,

$$\begin{aligned} & \mathbb{E}F(\bar{\zeta}^{n,[0,T]}) - \mathbb{E}F(\xi^{n,[0,T]}) \\ &= \underbrace{\mathbb{E}U_0^n(\bar{\zeta}_0^n) - \mathbb{E}U_0^n(\xi_*)}_{\text{initial Gaussian approximation}} \\ &+ \underbrace{\int_0^T \mathbb{E}[(\mathcal{L}_{\rho_s^n} - \mathcal{G}_{\varrho_s^n})U_s^n(\bar{\zeta}^{n,[0,s]})] ds}_{\text{accumulated generator mismatch}}. \end{aligned}$$

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A path-space weak error reduces to an initial comparison and a comparison of generators.

Both generators act on the current state while retaining the same past.

Generator comparison

The microscopic and modified Gaussian generators match up to

$$\mathcal{L}_\rho^n = \mathcal{G}_\varrho^n - \mathcal{C}^n + \text{remainder terms.}$$

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The **microscopic correlation correction** is

$$\mathcal{C}^n G(\bar{\zeta}) = \frac{1}{2n^d} \sum_{j=1}^d \left\langle (\partial_{n,j}^{(m_\partial)})^{\odot 2} D^2 G(\bar{\zeta}), \zeta \tau_{n,j} \zeta \right\rangle_n .$$

It couples the Hessian with products of neighbouring fluctuations.

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The remaining contributions to the weak error are controlled by

$$n^{-d/2} + n^{-2m_\Delta} + n^{-m_\partial} + n^{-(m_\tau+1)} .$$

Increasing the spatial orders removes the old n^{-1} error.

The correlation-corrected approximation

Write the weak error and the correlation contribution as

$$\mathcal{E}_n(F) := \mathbb{E}F(\bar{\zeta}^n, [0, T]) - \mathbb{E}F(\xi^n, [0, T]),$$

$$C_n(F) := \int_0^T \mathbb{E} \mathcal{C}^n U_t^n(\bar{\zeta}^n, [0, t]) dt.$$

The correlation-corrected approximation

Write the weak error and the correlation contribution as

$$\mathcal{E}_n(F) := \mathbb{E}F(\bar{\zeta}^{n,[0,T]}) - \mathbb{E}F(\xi^{n,[0,T]}),$$

$$C_n(F) := \int_0^T \mathbb{E} \mathcal{C}^n U_t^n(\bar{\zeta}^{n,[0,t]}) dt.$$

Theorem [Gess–K. '26+]

Under the assumptions of the functional CLT,

$$\mathcal{E}_n(F) = -C_n(F) + R_n(F),$$

$$|R_n(F)| \leq C(n^{-d/2} + n^{-\beta})\|F\|_{\mathcal{P}^3}, \quad \beta = \min\{2m_\Delta, m_\partial, m_\tau + 1\}.$$

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If $\beta \geq d/2$, retaining the correlation correction leaves a remainder of order $n^{-d/2}$.

Exclusion correlations

For a neighbouring pair $y = x + e_j/n$, the mean squared occupation difference is

$$\begin{aligned}\mathbb{E}\left(\eta_t^n(x) - \eta_t^n(y)\right)^2 &= \rho_t^n(x) + \rho_t^n(y) - 2\rho_t^n(x)\rho_t^n(y) \\ &\quad - 2\text{Cov}\left(\eta_t^n(x), \eta_t^n(y)\right).\end{aligned}$$

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The correction $\mathcal{C}^n$ retains neighbouring products of the centered occupations. Correlation estimates give

$$|C_n(F)| \lesssim n^{-\min\{d/2, 2\}} \|F\|_{\mathcal{P}^3}.$$

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Thus, for $\beta \geq d/2$,

$$|\mathcal{E}_n(F)| \lesssim n^{-\min\{d/2, 2\}} \|F\|_{\mathcal{P}^3}.$$

For $d > 4$, the correlation term determines this upper bound.

A quadratic observable detects the error

Consider the variance observable

$$Q(\gamma) = \frac{1}{2} \langle \gamma_{t_0}, \varphi \rangle^2.$$

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Quadratic weak error [Gess–K. '26+]

For a nonconstant initial profile, $d \geq 3$ and $\beta > 2$, there are smooth φ and $t_0 \in (0, T]$ such that

$$c n^{-2} \leq \left| \mathbb{E} Q(\bar{\zeta}^{n, [0, T]}) - \mathbb{E} Q(\xi^{n, [0, T]}) \right| \leq C n^{-2}$$

for all sufficiently large n .

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For this quadratic observable, $R_n(Q) = O(n^{-\beta})$. The n^{-2} contribution therefore comes from microscopic correlations.

For $d > 4$, this explains the n^{-2} barrier for the chosen Gaussian model.

Conclusions

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$$n^{-\min\{d/2, 1\}} \longrightarrow n^{-\min\{d/2, 2\}}.$$

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2. **The error has a microscopic correlation contribution.** Keeping it explicit leaves an $O(n^{-d/2})$ remainder at sufficiently high spatial order. A quadratic observable detects the n^{-2} scale.
3. **A path-space comparison method.**

backward equation + martingale formula $\implies$ generator comparison.

References

- [1] Benjamin Gess and Vitalii Konarovskiy.
A Quantitative Central Limit Theorem for the Simple Symmetric Exclusion Process,
to appear in Ann. Prob.
- [2] Benjamin Gess and Vitalii Konarovskiy.
Higher-Order Gaussian Approximations on Path Space for the Simple Symmetric Exclusion Process
(2026+)

Thank you!