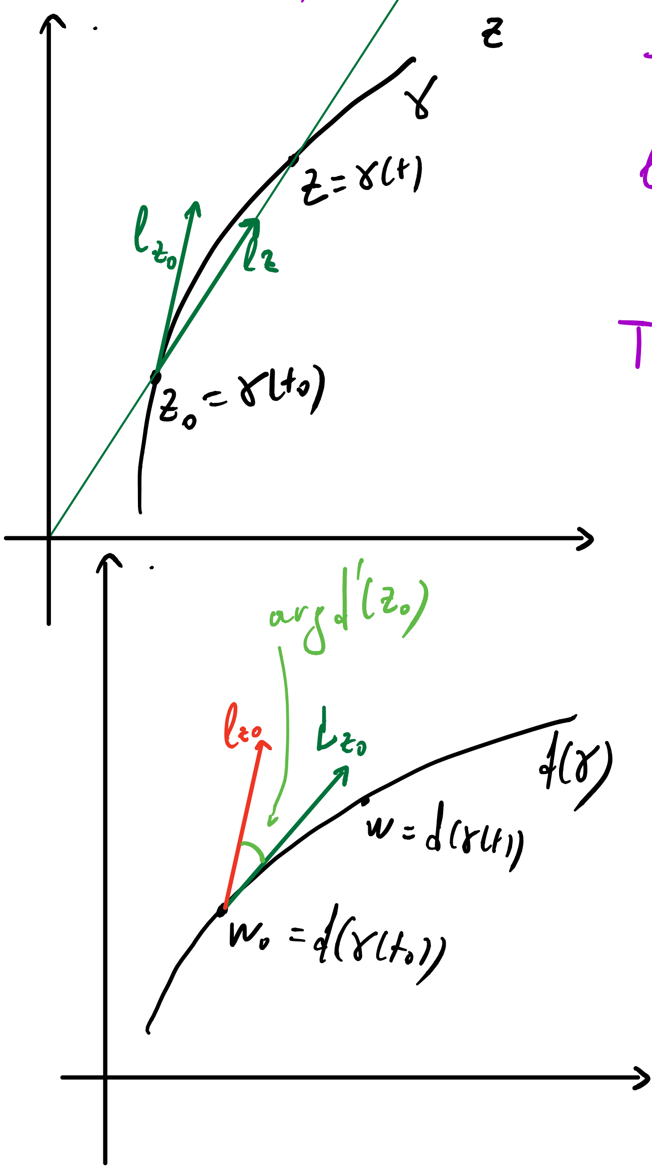
16 Conformal maps

1. Geometric meaning of $\arg f'(z)$ and $|f'(z)|$

Let $\gamma(t) = x(t) + i\,y(t)$, $t \in [\alpha, \beta]$ be a continuous path in $\mathbb{C}$ ($x(t)$, $y(t)$, $t \in [\alpha,\beta]$ are continuous functions). We also assume that γ is continuously differentiable.

Let also $f : U \to \mathbb{C}$ be function with $f'(z_0) \neq 0$. Denote $w_0 = f(z_0)$.

Assume, $\gamma(t_0) = z_0$.



Set

$$l_z := \frac{z - z_0}{|z - z_0|} = \frac{\gamma(t) - \gamma(t_0)}{|\gamma(t) - \gamma(t_0)|}$$

Then

$$l_{z_0} = \lim_{z \to z_0} \frac{z - z_0}{|z - z_0|} = \frac{\gamma'(t)}{|\gamma'(t)|}$$

can be identified with unit tangent vector to γ at z_0. Next, we consider the image of γ under the map f

Find the tangent vector to $f(\gamma)$ at w_0.

Similarly,

$$L_{z_0} = \lim_{w \to w_0} \frac{w - w_0}{|w - w_0|} = \lim_{t \to t_0} \frac{f(\gamma(t)) - f(\gamma(t_0))}{|f(\gamma(t)) - f(\gamma(t_0))|} =$$

$$= \lim_{t \to t_0} \frac{f(\gamma(t)) - f(\gamma(t_0))}{\gamma(t) - \gamma(t_0)} \cdot \frac{\gamma(t) - \gamma(t_0)}{|\gamma(t) - \gamma(t_0)|} \cdot \frac{|\gamma(t) - \gamma(t_0)|}{|f(\gamma(t)) - f(\gamma(t_0))|} =$$

$$= f'(z_0) \cdot \ell_{z_0} \cdot \frac{1}{|f'(z_0)|}$$

So, $L_{z_0} = \dfrac{f'(z_0)}{|f'(z_0)|} \ell_{z_0}$

Next, we compute

$$\arg L_{z_0} = \arg \frac{f'(z_0)}{|f'(z_0)|} \ell_{z_0} =$$

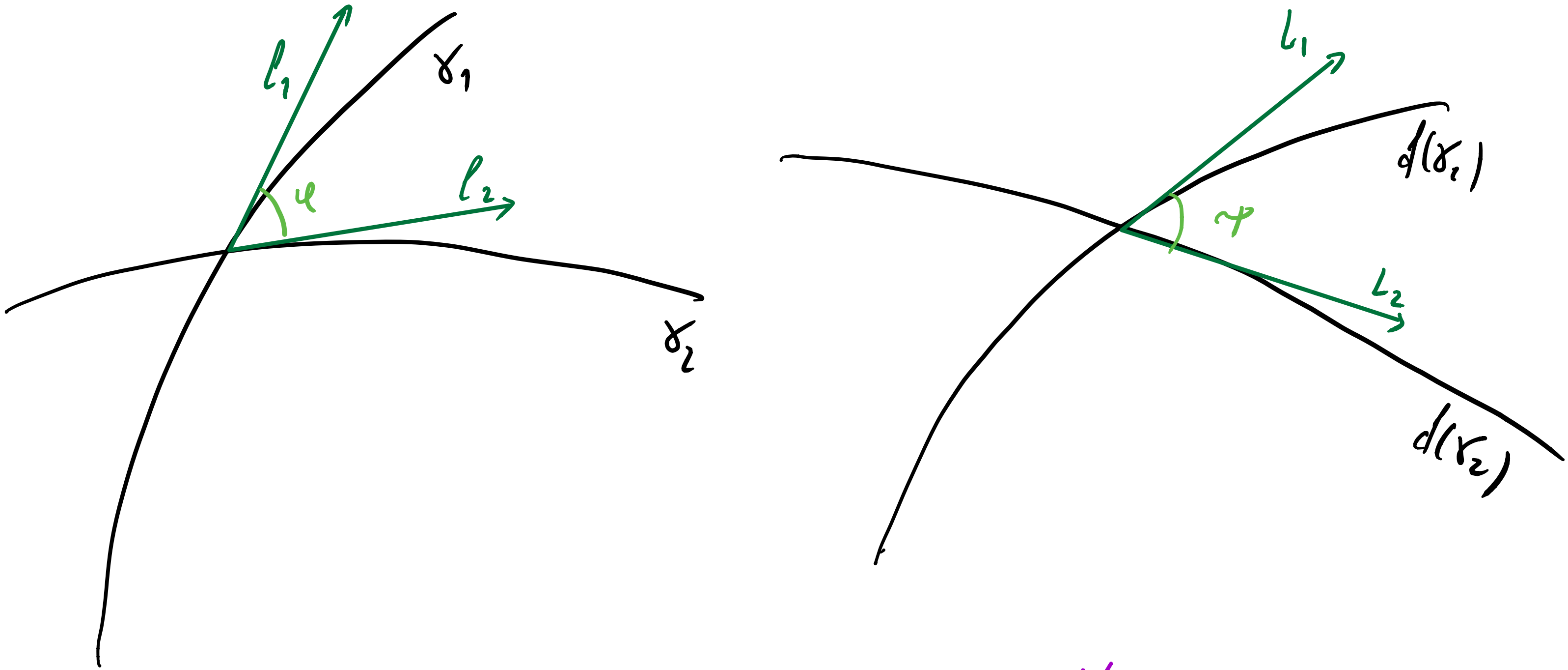
$$= \arg f'(z_0) + \arg \ell_{z_0} - \underbrace{\arg |f'(z_0)|}_{=0}$$

Hence,

$$\arg L_{z_0} = \arg f'(z_0) + \arg \ell_{z_0}.$$

Under the map f a tangent line to any curve at z_0 is rotated on the angle $\arg f'(z_0)$

Let us consider two paths through z_0. The angle between this two paths is defined as the angle between their tangent vectors.



Then $\phi = \arg L_2 - \arg L_1 = \arg f'(z_0) + \arg l_2 -$
$- \arg f'(z_0) - \arg l_1 = \arg l_2 - \arg l_1 = \varphi$.

Corollary 16.1 If $f'(z_0) \neq 0$, then the function f preserves the angles between curves which pass through z_0.

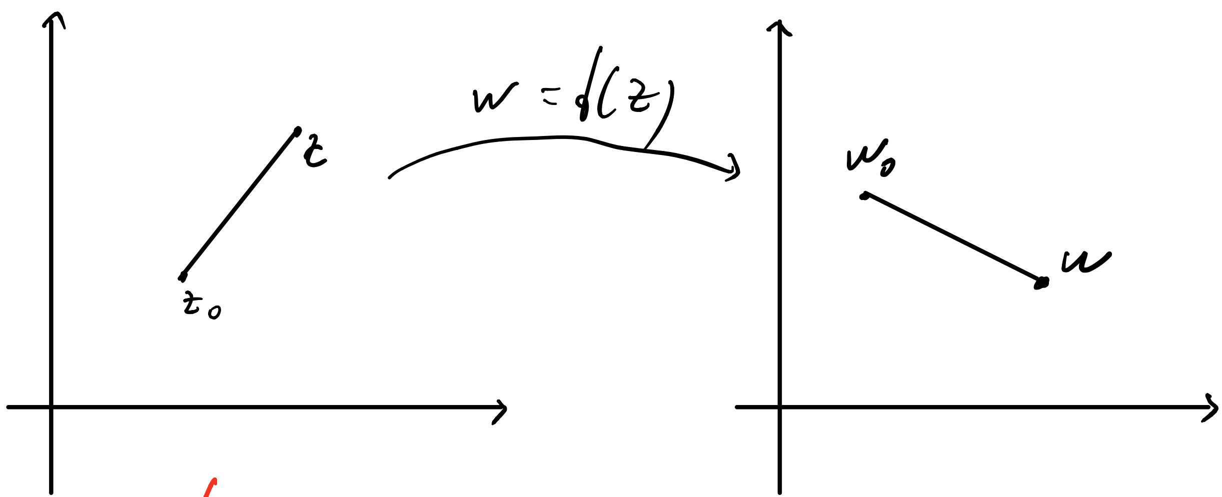
Def 16.1 a) A continuous map $f: U \to \mathbb{C}$ which preserves the angles beetween curves which path through $z_0 \in U$ is called **conformal at z_0**.
b) If f is conformall at any point of U then f is called **conformal on U**.

Theorem 16.1 A holomorphic function f is conformal at any point where its derivative is non-zero.

Next, we explain the geometric meaning of $|f'(z_0)|$.

We write

$$|f'(z_0)| = \lim_{z\to z_0}\left|\frac{f(z)-f(z_0)}{z-z_0}\right| = \lim_{z\to z_0}\frac{|w-w_0|}{|z-z_0|}$$



So $|f'(z_0)|$ equals to the dilation coefficient at z_0 under the mapping f.

2. Fractional linear transformations

a) Conformal property on $\overline{\mathbb{C}}$.

Fractional linear transformations are functions of the form

$$w = \frac{az+b}{cz+d}, \qquad ad-bc=0, \qquad (16.1)$$

where a, b, c are fixed complex numbers, and z is the complex variable. The condition $ad-bc$ is imposed to exclude the degenerate case when $w = \text{const}$.

The function (16.1) is defined for all $z \neq -\frac{d}{c}$ (if $c \neq 0$). We set $w = \infty$ at $z = -\frac{d}{c}$

Theorem 16.2 A fractional linear transformation (16.1) is a homeomorphism (that is, a continuous and one-to-one map) of $\bar{\mathbb{C}}$ onto $\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$

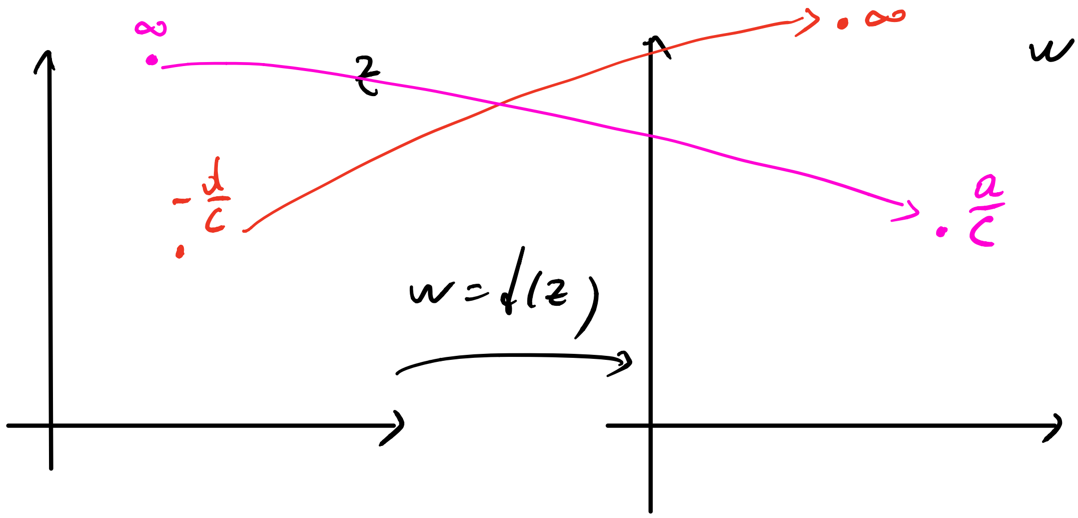
Proof We assume $c \neq 0$ (case $c = 0$ is trivial)

The function $w(z)$ is defined everywhere in $\bar{\mathbb{C}}$ and

$$z = \frac{dw - b}{a - cw}, \quad w \neq \frac{a}{c}, \infty$$

has exactly one pre-image. Moreover, the extension of $w(z)$ to $\bar{\mathbb{C}}$ shows that

$$\infty = w\left(-\frac{d}{c}\right) \text{ and } \frac{a}{c} = w(\infty)$$



It remains to show the continuity of (16.1). The continuity is obvious at $z \neq -\frac{d}{c}, \infty$, and

$$\lim_{z \to -\frac{d}{c}} \frac{az+b}{cz+d} = \infty, \quad \lim_{z \to \infty} \frac{az+b}{cz+d} = \frac{a}{c}.$$

Def 16.2 Let γ_1 and γ_2 be two paths that pass through the point $z=\infty$. The angle between γ_1, γ_2 at $z=\infty$ is the angle between their images Γ_1, Γ_2 under the map $z \mapsto \frac{1}{z}$ at the point 0.

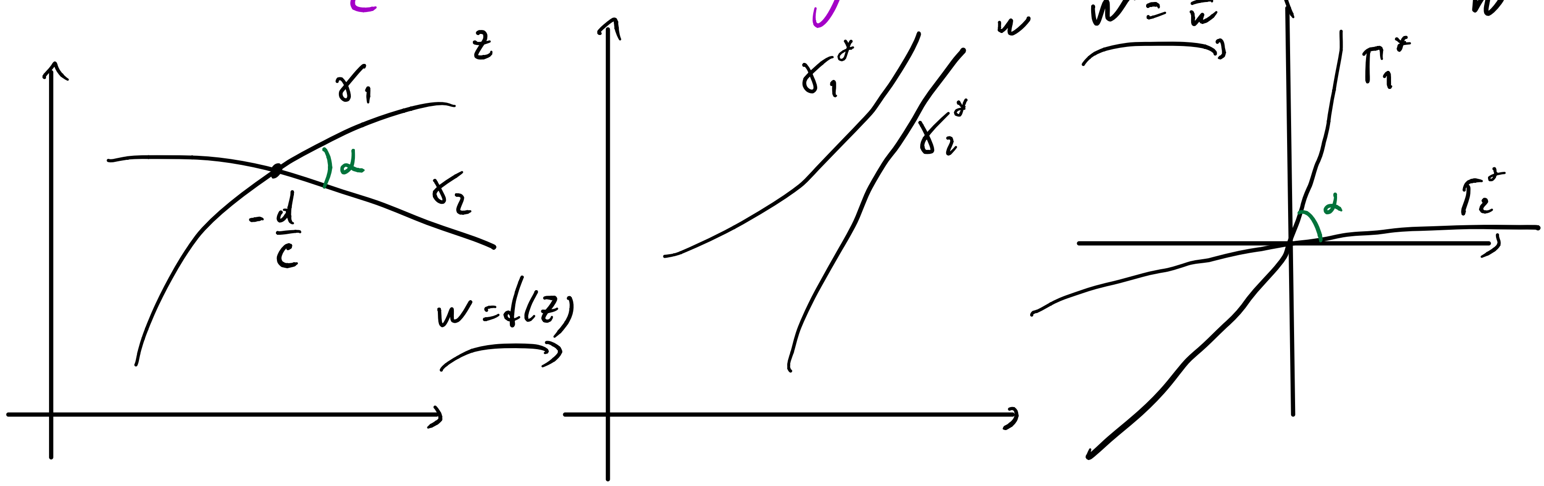
Theorem A fractional linear map is conformal on $\bar{\mathbb{C}}$.

Proof We note that for all $z \in \bar{\mathbb{C}} \setminus \{-\frac{d}{c}, \infty\}$

$$w'(z) = \frac{a(cz+d)-(az+b)c}{(cz+d)^2} = \frac{ad-bc}{(cz+d)^2} \neq 0.$$

So, map (16.1) is conformal at any point of $\bar{\mathbb{C}} \setminus \{-\frac{d}{c}, \infty\}$.

Next, let γ_1, γ_2 be two paths intersecting at $z = -\frac{d}{c}$ at an angle α.



The angle between their images γ_1^* and γ_1^* is the angle between the images Γ_1^* and Γ_1^* of γ_1^* and γ_2^* under the map $W = \frac{1}{w}$ at the point $W = 0$. Note

$$W(z) = \frac{cz + d}{az + b},$$

so Γ_1^*, Γ_2^* are the images of γ_1 and γ_2 under this map. Similarly,

$$W'(z) = \frac{bc - ad}{(az+b)^2} \neq 0 \quad \text{at } z = -\frac{d}{c}.$$

Hence the angle between Γ_1^* and Γ_2^* at $W = 0$ is equal to α.

The case $z = \infty$ can be proved similarly. ∎

2. Geometric properties

We first introduce the convention that a circle in $\overline{\mathbb{C}}$ is either a circle or a straight line on the complex plane $\mathbb{C}$

Theorem 16.3 Fractional linear transformations map a circle in $\overline{\mathbb{C}}$ onto a circle in $\overline{\mathbb{C}}$

Proof The statement is trivial for $c = 0$

since linear transformations are a composition of a shift, rotation and dilation that all have the property stated in the theorem.

If $c \neq 0$ then the mapping

$$L(z) = \frac{az+b}{cz+d} = \frac{a}{c} + \frac{bc-ad}{c(cz+d)} = A + \frac{B}{z+C}$$

Therefore L is a composition

$$L = L_1 \circ L_2 \circ L_3 \quad \text{of}$$

$$L_1(z) = A + Bz, \quad L_2(z) = \frac{1}{z}, \quad L_3(z) = z + C.$$

L_1 (dilation with rotatoon) and L_3 (shift) map circles in $\overline{\mathbb{C}}$ onto circles in $\overline{\mathbb{C}}$.

It remains to prove this property for the map

$$L_2(z) = \frac{1}{z}.$$

Any circle in $\overline{\mathbb{C}}$ may be represented as

$$E(x^2+y^2) + F_1 x + F_2 y + C = 0. \qquad (16.2)$$

Using the complex variables $z = x+iy$, $\bar{z} = x - iy$, that is $x = \frac{z+\bar{z}}{2}$, $y = \frac{1}{2i}(z-\bar{z})$ we may rewrite (16.2) as follows

$$E z\bar{z} + Fz + \bar{F}\bar{z} + G = 0 \qquad (16.3)$$

with $F = \frac{F_1 - iF_2}{2}$, $\bar{F} = \frac{F_1 + iF_2}{2}$.

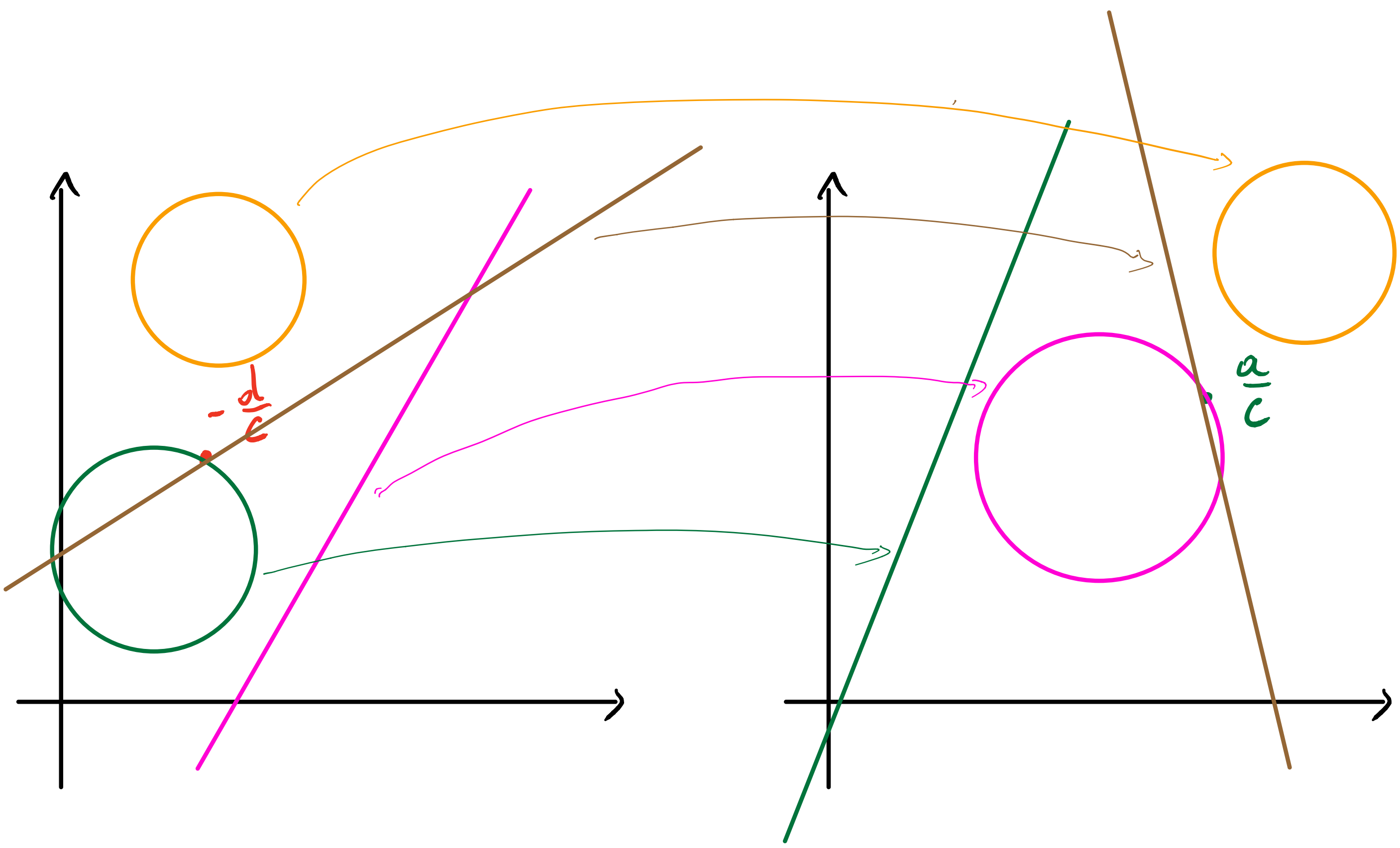
in order to obtain the equation for the image of the circle (16.3) under L_2 it suffices to set $z = \frac{1}{w}$ in (16.3) to get

$$E + F\bar{w} + \bar{F}w + G w\bar{w} = 0.$$

Remark 16.1 Let ℓ be a circle in $\bar{\mathbb{C}}$ and L its image under the map (16.1).

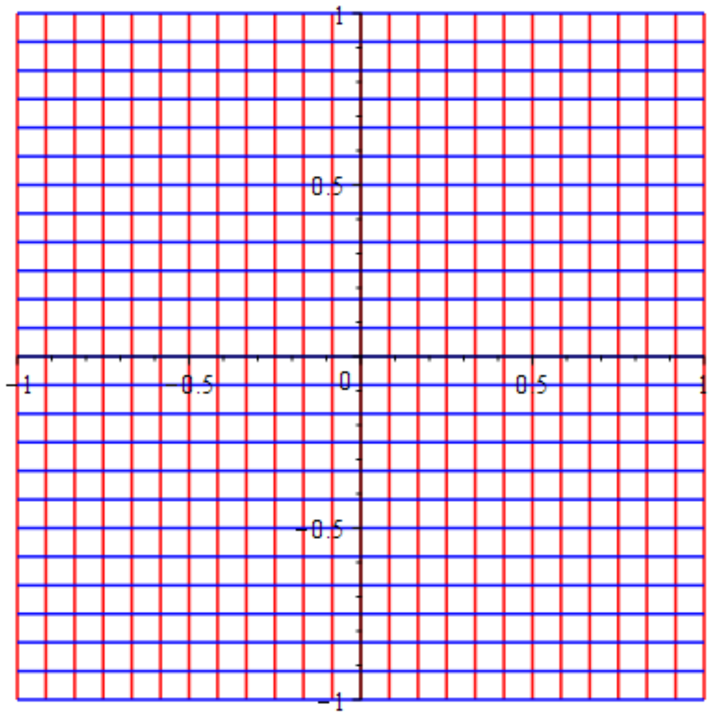
a) $-\frac{d}{c} \in \ell \iff L$ is a straight line

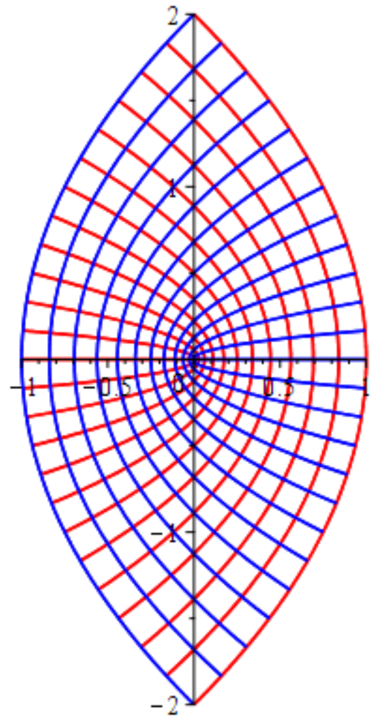
b) ℓ is a straight line $\iff \frac{a}{c} \in L$.



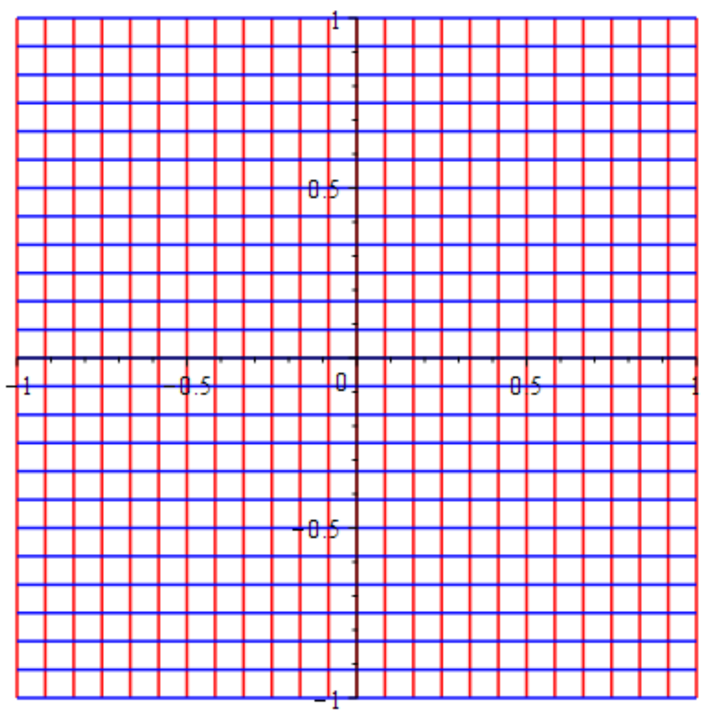
Some simulations

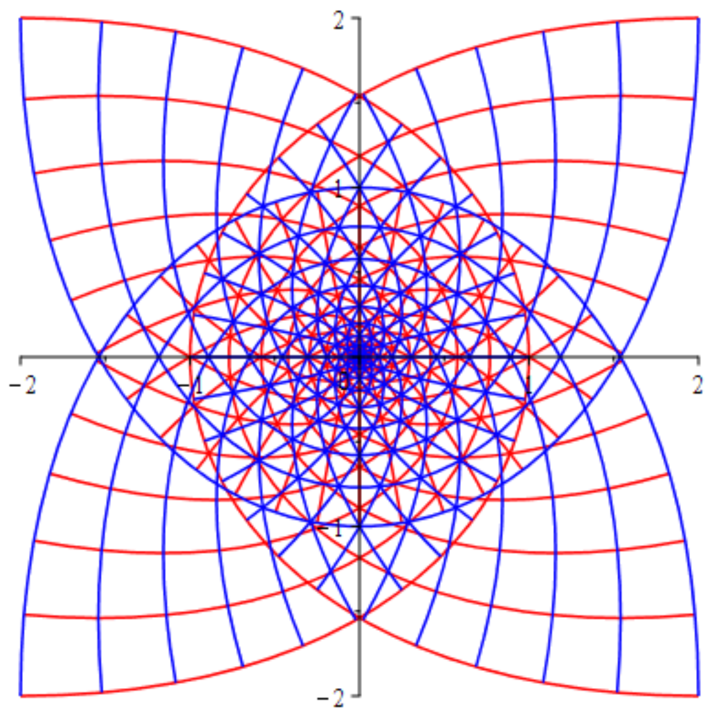
$$f(z) = z^2$$



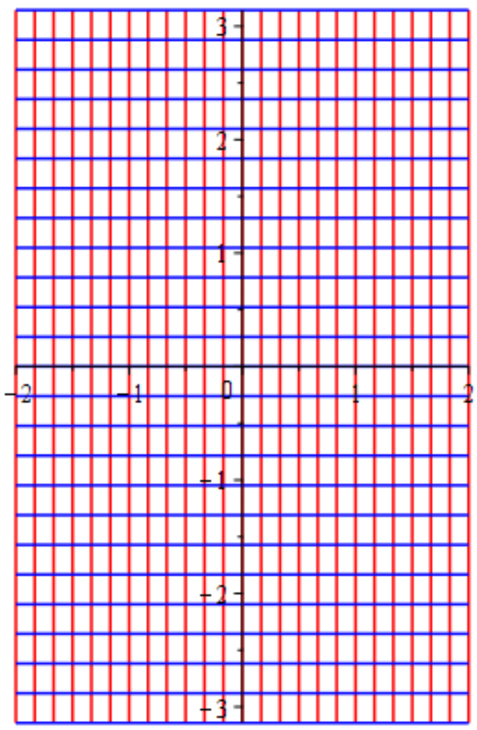


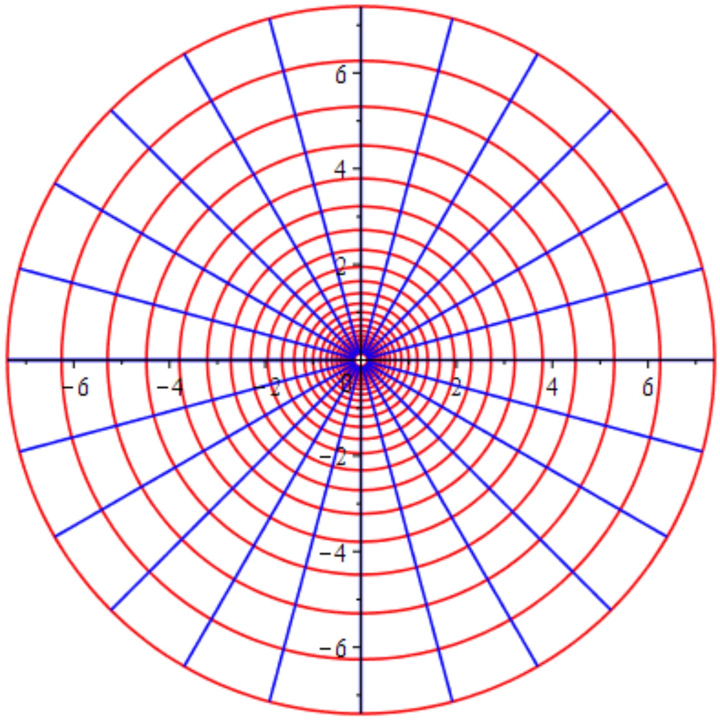
$$f(z) = z^3$$



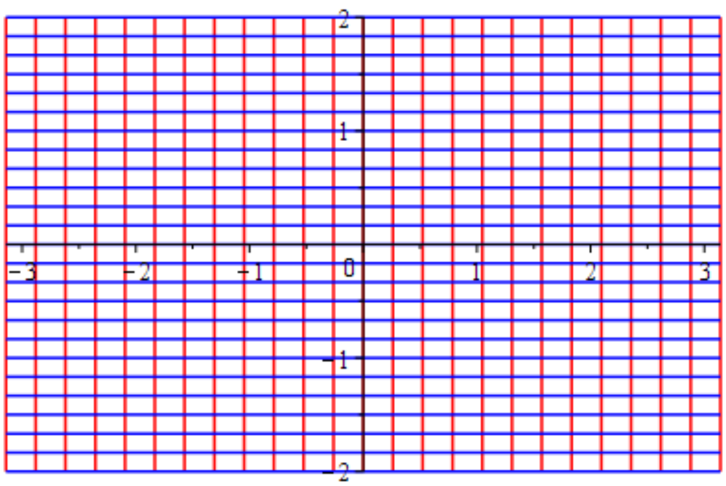


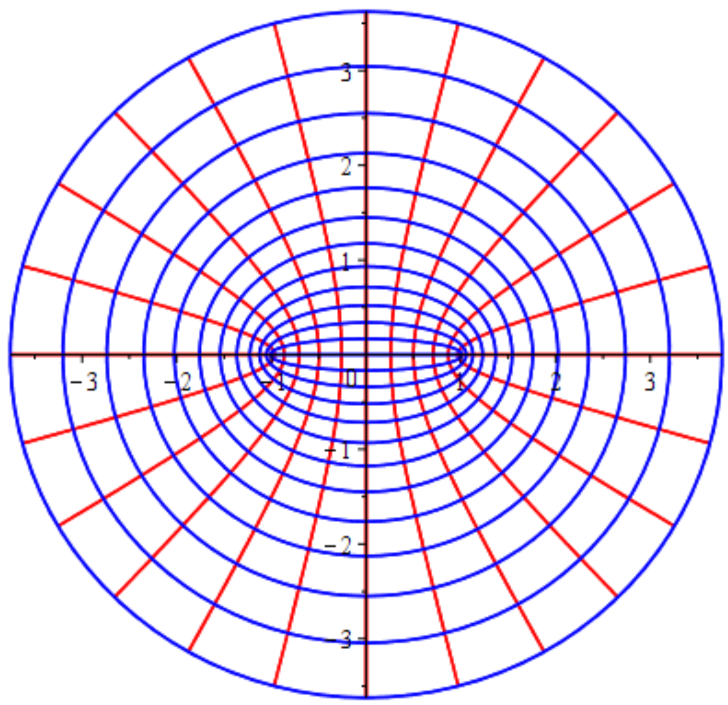
$$f(z) = e^z$$



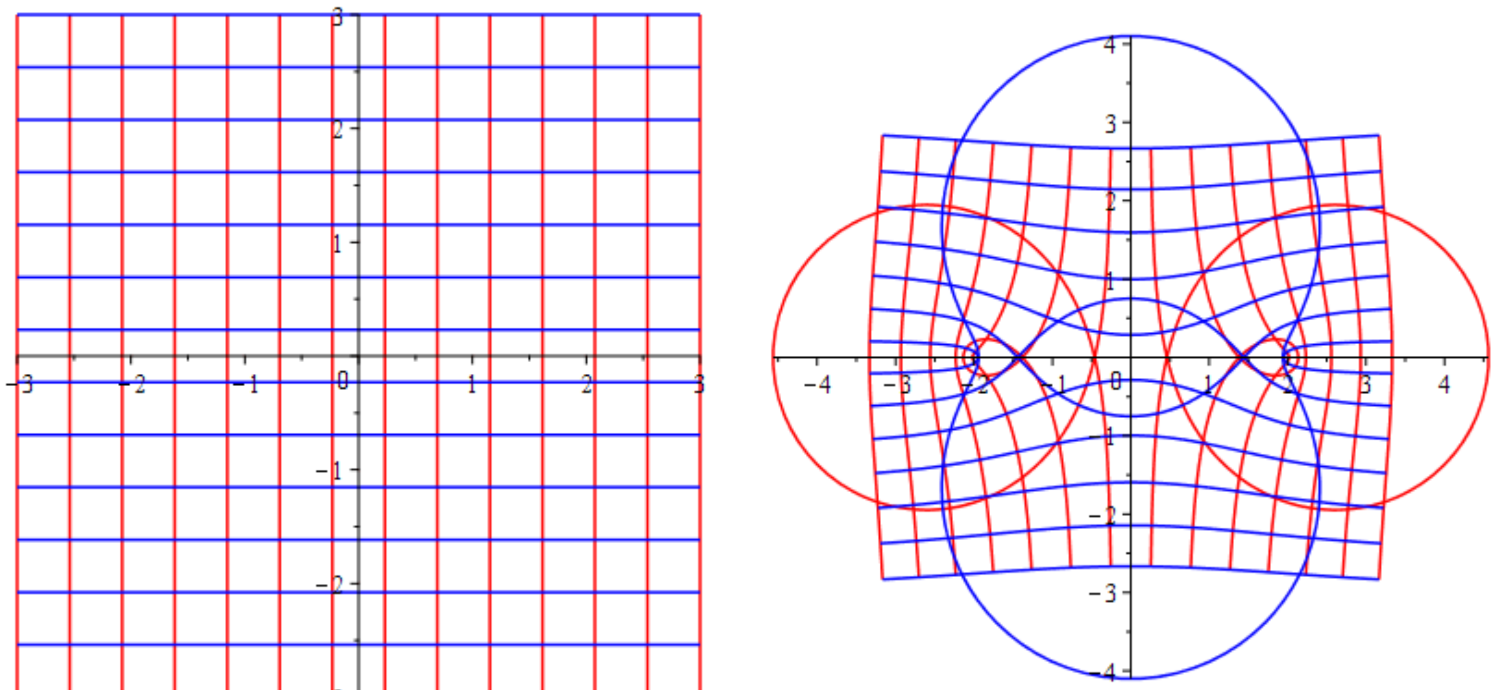


$$f(z) = \sin z$$





$$f(z) = z + \frac{1}{z}$$



$$f(z) = \frac{z}{z-1}$$