23. Compact linear operators

1. Definition and properties of compact linear operators on normed spaces.

Let X be a normed space. We first recall that a set $F \subset X$ is compact if every open cover of F contains a finite subcover, that is, for every family of open sets $\{G_\alpha\}$ such that $F \subset \bigcup_\alpha G_\alpha$ there exists $\{G_{\alpha_1}, \dots G_{\alpha_n}\}$ such that

$$F \subset \bigcup_{k=1}^{n} G_{\alpha_k}.$$

Th 23.1 F is compact in X iff every sequence $\{x_n\}_{n\geq 1} \subset F$ has a convergent in F subsequence, that is $\exists \{x_{n_k}\}_{k\geq 1}$ such that $x_{n_k} \to x \in F$.

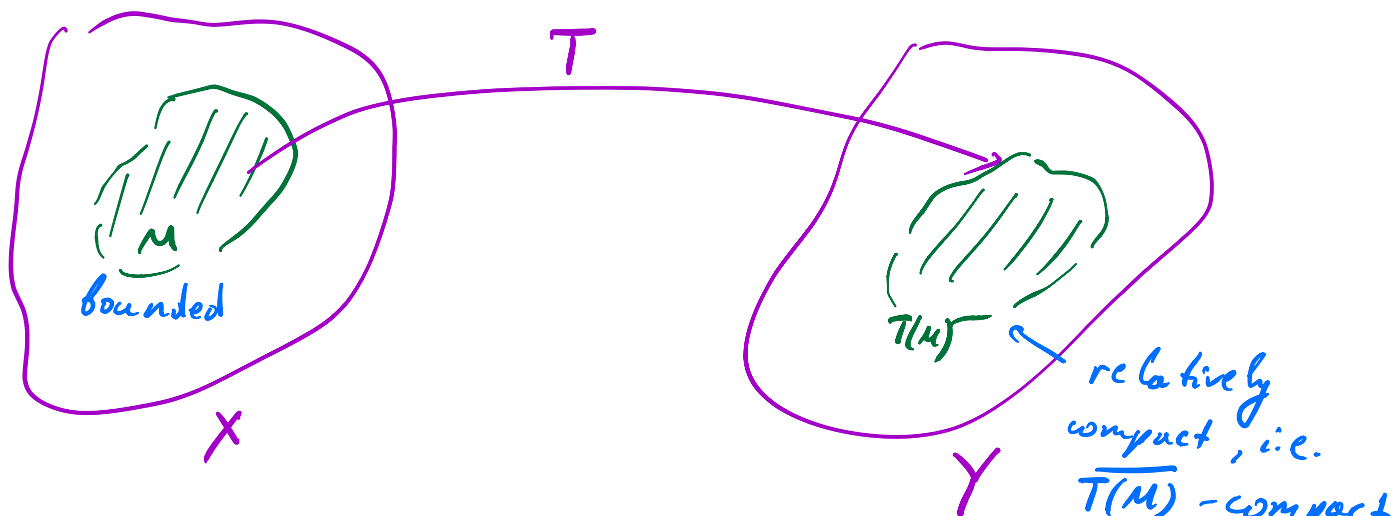
Def. 23.2 A set $F \subset X$ is called *relatively compact* if $\overline{F}$ is compact

Every bounded set in a finite-dimensional normed space is relatively compact.

Exercise 23.3 Show that F is relatively compact if and only if $\forall \{x_n\}_{n\geq 1} \subset F$ there exists a subsequence $\{x_{n_k}\}_{k\geq 1}$

such that $x_{n_k} \to x$ (where not necessarily $x \in F$).

Def 23.4 Let X, Y be normed spaces. An operator $T: X \to Y$ is called a **compact linear operator** if T is linear and if for every bounded subset $M \subset X$, the image $T(M)$ is relatively compact.



Since every compact set is bounded.
So, every compact operator is bounded because the image of the sphere $\{x : \|x\| = 1\}$ is relatively compact and hence, bounded.

Th 23.5 (Compactness criterion) Let X and Y be normed space and $T: X \to Y$ be a linear operator. Then T is compact iff it maps every bounded sequene $\{x_n\}_{n\geq 1}$ in X onto a sequence $\{Tx_n\}$ in Y which has a convergent subsequence, that is,

$\forall \{x_n\}_{n\geq 1}$ - bounded $(\exists C : \|x_n\| \leq C \ \forall n \geq 1)$

$\Rightarrow$ there exists subsequence $\{Tx_{n_k}\}_{k\geq 1}$ of $\{Tx_n\}_{n\geq 1}$ s.t.

$$Tx_{n_k} \to y \quad \text{in } Y.$$

Th 23.6. If $T: X \to Y$ be bounded and $\operatorname{Im} T = T(X)$ is finite dimensional, then T is compact

Example 23.7. Take $X = Y = \ell^2$ over field K

$$Tx = (2\xi_1, \xi_2, \xi_3 + \xi_4, 0, 0, 0, \dots)$$

The operator T is compact. Indeed,

$T(X) = \{(\eta_1, \eta_2, \eta_3, 0, 0, \dots) : \eta_1, \eta_2, \eta_3 \in K\}$

– 3-dim. subspace of ℓ^2.

By Th. 23.6 Tx is compact. ●

Th 23.8. Let $\{T_n\}_{n\geq 1}$ be a sequence of compact linear operators from a normed space X into a Banach space Y and $T_n \to T$ in $B(X,Y)$ then T is compact.

Example 23.9. We consider $X = Y = \ell_2$ and

$$Tx = \left(\xi_1, \frac{\xi_2}{2}, \frac{\xi_3}{3}, \dots\right)$$

Let us prove that T is compact.
Take

$$T_n x = \left(\xi_1, \frac{\xi_2}{2}, \frac{\xi_3}{3}, \dots, \frac{\xi_n}{n}, 0, 0, \dots\right).$$

Then T_n is bounded and $\dim(T_n(X)) = n$.
So, by Th. 23.6 it is compact.
Let us compute $\|T - T_n\|$.

$$\|(T-T_n)x\|^2 = \left\|\left(0, 0, \dots, 0, \frac{\xi_{n+1}}{n+1}, \frac{\xi_{n+2}}{n+2}, \dots\right)\right\|^2 =$$

$$= \sum_{k=n+1}^{\infty} \frac{\xi_k^2}{(n+1)^2} \leq \frac{1}{(n+1)^2} \sum_{k=n+1}^{\infty} \xi_k^2 \leq \frac{1}{(n+1)^2}\|x\|^2.$$

Hence, $\|T - T_n\| \leq \frac{1}{n+1} \to 0, \quad n \to \infty.$

By Th 23.8, T is compact.

2. Spectral properties of compact self-adjoint operators.

In this section, we will assume that H is a separable Hilbert space.

Th 23.10 Let $T: H \to H$ be a bounded linear operator. The following statements are equivalent.

a) T is compact

b) T^* is compact

c) if $\langle x_n, y \rangle \to \langle x, y \rangle$ $\forall y \in H$, then

$$T x_n \to T x \text{ in } H.$$

d) there exists a sequence T_n of operators of finite rank (i.e. $\dim T(H) < \infty$) such that

$$\| T - T_n \| \to 0.$$

Th 23.11 (Hilbert – Schmidt theorem)

Let T be a self-adjoint compact operator. Then

(i) There exist an orthonormal basis consisting of eigenvectors of T

(ii) All eigenvalues of T are real and for every eigenvalue $\lambda \neq 0$ the corresponding eigenspace is finite dimensional

(iii) Two eigenvalues of T that correspond to different eigenvalues are orthogonal

(iv) If T has countable set of eigenvalues (not finite set) $\{\lambda_n\}_{n\geq 1}$, then

$$\lambda_n \to 0, \quad n \to \infty.$$

Corollary 23.12. Let T be a compact self-adjoint linear operator on a complex Hilbert space H. There exists an orthonormal basis $\{e_k\}_{k\geq 1}$ such that

$$Tx = \sum_{n=1}^{\infty} \lambda_n \langle x, e_n \rangle e_n, \quad x \in H.$$