24 Unbounded linear operators

1. Examples of unbounded linear operators.

We start from the following examples.
Let $H = L^2(-\infty, +\infty)$

Multiplication operator:

$$(Tx)(t) = t\,x(t), \quad t \in \mathbb{R}$$

$$\mathcal{D}(T) = \left\{ x \in L^2(-\infty, +\infty) : \int_{-\infty}^{+\infty} t^2 |x(t)|^2 dt < \infty \right\}$$

We remark that $\mathcal{D}(T) \neq L^2(-\infty, +\infty)$.
Indeed,

$$x(t) = \begin{cases} \frac{1}{t}, & t \geq 1 \\ 0, & t < 1. \end{cases} \in L^2(-\infty, +\infty)$$

because

$$\|x\|^2 = \int_{-\infty}^{+\infty} |x(t)|^2 dt = \int_{1}^{+\infty} \frac{1}{t^2}\, dt = -\frac{1}{t}\Big|_1^{+\infty} = 1,$$

but

$$\|Tx\|^2 = \int_{-\infty}^{+\infty} t^2 |x(t)|^2 dt = \int_1^{+\infty} 1\, dt = +\infty.$$

The multiplication operator is unbounded.
Let us recall that a linear operator
$T: D(T) \to H$ is bounded if

$$\exists\, C \geq 0 \quad \|Tx\| \leq C\|x\|, \quad x \in D(T)$$

We take

$$x_n = \begin{cases} 1, & n \leq t < n+1 \\ 0, & \text{otherwise}, \end{cases}$$

then $\|x_n\|^2 = \int_{-\infty}^{+\infty} |x_n(t)|^2\, dt = \int_n^{n+1} dt = 1$

But

$$\|Tx_n\|^2 = \int_{-\infty}^{+\infty} t^2 |x_n(t)|^2\, dt =$$

$$= \int_n^{n+1} t^2\, dt \geq n^2$$

So, $\|Tx_n\|^2 \geq n^2 \|x_n\| \quad \forall n \geq 1.$

Hence, T is unbounded.

Differentiation operator

$$Tx(t) = i\frac{d}{dt}x(t),$$

$\mathcal{D}(T) \subset L^2(-\infty, +\infty)$.

Later we will explain what $\mathcal{D}(T)$ is. Here we only remark that all continuously differentiable functions with compact support and Hermite polynomials belong to $\mathcal{D}(T)$

Taking

2. Symmetric and self-adjoint linear operators.

Let H be a complex Hilbert space. Let $T : \mathcal{D}(T) \to H$ be a densely defined (i.e., $\mathcal{D}(T)$ is dense in H) linear operator. The adjoint operator

$$T^* : \mathcal{D}(T^*) \to H$$

of T is defined as follows. The domain $\mathcal{D}(T^*)$ of T^* consists of all $y \in H$ such that $\exists\, y^* \in H$ satisfying

$$\langle Tx, y \rangle = \langle x, y^* \rangle \quad \forall x \in \mathcal{D}(T).$$

For each such $y \in \mathcal{D}(T^*)$ define

$$T^* y := y^*.$$

Since $D(T)$ is dence, y^* is uniquely defined.

Before, we discuss properties of adjoint operators, we discuss the extension of a linear operator.

Let us come back to the operator

$$(T_1 x)(t) = i x'(t).$$

we can define T_1 only on functions

$$D(T_1) = C_0^1(\mathbb{R})$$

$$= \{ f \in C^1(\mathbb{R}) : f = 0 \text{ outside some interval} \}$$

Now, let

$$(T_2 x)(t) = i x'(t),$$

$$D(T_2) = \left\{ f \in C(\mathbb{R}) : \int_{-\infty}^{+\infty} |f|^2 dt < \infty, \int_{-\infty}^{+\infty} |f'|^2 dt < +\infty \right\}$$

They are different operators, but

$D(T_1) \subset D(T_2)$ and $T_1 = T_2|_{D(T_1)}$.

Def. 24.1 An operator T_2 is called an **extension** of an operator T_1 if

$$D(T_1) \subset D(T_2) \text{ and } T_1 = T_2|_{D(T_1)}.$$

We will use the notation

$$T_1 \subset T_2.$$

Th 24.2 Let $S : D(S) \to H$ and $T : D(T) \to H$ be densely defined linear operators. Then

a) if $S \subset T$, then $T^* \subset S^*$

b) if $D(T^*)$ is dence in H, then $T \subset T^{**}$.

c) if T is injective and $\operatorname{Im} T$ is dence in H, then T^* is injective and

$$(T^*)^{-1} = (T^{-1})^*.$$

Def 24.3 Let $T : D(T) \to H$ be densely defined linear operator on H. T is called a **symmetric linear operator** if

$$\langle Tx, y \rangle = \langle x, Ty \rangle \quad \forall x, y \in D(T).$$

Remark that if T is symetric it does not imply that $T = T^*$. Indeed, take take

$$(Tx)(t) = i\,x(t),$$

$$D(T) = C_0(\mathbb{R}).$$

Then

$$\langle Tx, y\rangle = \int_{-\infty}^{+\infty} i\,x'(t)\,\overline{y(t)}\,dt =$$

$$= \int_{-\infty}^{+\infty} i\,\overline{y(t)}\,dx(t) = i\overline{y(t)}x(t)\Big|_{-\infty}^{+\infty} -$$

$$- \int_{-\infty}^{+\infty} x(t)\,d\left(i\overline{y(t)}\right) =$$

$$= 0 - 0 - \int_{-\infty}^{+\infty} x(t)\,i\,\overline{y'(t)}\,dt =$$

$$= \int_{-\infty}^{+\infty} x(t)\,\overline{i y'(t)}\,dt = \langle x, Ty\rangle,$$

$\forall\ x, y \in D(T) = C_0(\mathbb{R})$.

However, $T^* \neq T$. For instance $y(t) = e^{-t^2}$, $t \in \mathbb{R}$, does not belong to $\mathcal{D}(T) = C_0(\mathbb{R})$, but $y \in \mathcal{D}(T^*)$, because for $y^*(t) = i(-2t)e^{-t^2}$ one has

$$\langle Tx, y \rangle = \langle x, y^* \rangle$$

$\forall x \in \mathcal{D}(T)$. (check this!)

Lemma 24.4 A densely defined linear operator T is symmetric iff

$$T \subset T^*.$$

Def. 24.5. Let $T : \mathcal{D}(T) \to H$ be densely defined linear operator. T is called a *self-adjoint* if

$$T = T^*$$

Remark 24.6. Every self-adjoint operator is symmetric but not every symmetric operator is self-adjoint.

3. Closed linear operators

Def 24.7. Let $T : \mathcal{D}(T) \to H$ be a linear operator, where $\mathcal{D}(T) \subset H$. T is called a closed linear operator if its graph

$$Gr(T) = \{(x,y) : x \in D(T), y = Tx\}$$

is closed in $H \times H$, where the norm on $H \times H$ is defined by

$$\|(x,y)\| = (\|x\|^2 + \|y\|^2)^{\frac{1}{2}}.$$

Th 24.8. Let $T : D(T) \to H$ be a linear operator, where $D(T) \subset H$. Then

a) T is closed iff

$$\left.\begin{array}{l} x_n \to x, \quad x_n \in D(T) \\ Tx_n \to y \end{array}\right\} \Rightarrow \begin{array}{l} x \in D(T) \text{ and} \\ Tx = y. \end{array}$$

b) If T is closed and $D(T)$ is closed, then T is bounded.

c) Let T be bounded. Then T is closed iff $D(T)$ is closed.

Exercise 24.8. Show that the multiplication operator is closed.

Th 24.10 Let T be a densely defined operator on H. Then the adjoint operator T^* is closed.

Def. 24.11. If a linear operator T has an extension T_1 which is closed linear operator then T is called closable.

- If T is closable, then there exists a minimal closer oper. $\overline{T}$ satisfying $T \subset \overline{T}$. $\overline{T}$ is called the closure of T.

Def 24.12. Let $T: \mathcal{D}(T) \to H$ be a densely defined linear operator. If T is symmetric, its closure $\overline{T}$ exists (and is unique)