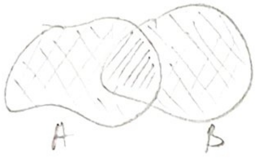
(2). Let $H$ be a ring. Show that $H$ is a semiring.
[2p.]

▶ $H$ is a ring: (i) $\forall A, B \in H: A \cup B \in H$;
(ii) $\forall A, B \in H: A \setminus B \in H$.

We need to show that $H$ is a semiring.
Check that 1) $A \cap B \in H$;
2) $A \setminus B = \bigcup_{k=1}^{n} C_k$, where $C_k \in H$, $C_k \cap C_j = \emptyset$, $k \neq j$.

1) $A \cap B = \underbrace{(A \cup B)}_{\in H, \text{ by (i)}} \setminus \underbrace{(\underbrace{(A \setminus B)}_{\in H} \cup \underbrace{(B \setminus A)}_{\in H})}_{\in H} \in H$. !




2) $A \setminus B = \underbrace{A \setminus B}_{= C_1 \in H \text{ by (ii)}}$.

Thus, $H$ is a semiring. ◀

(1) Prove that a nonempty class of sets $H \subset 2^X$ is a ring if only if $H$ is a semiring und $A \cup B \in H$ for every $A, B \in H$.

▶ We need to prove that $H$ - ring $\Leftrightarrow$ 1) $H$ is a semiring;
2) $\forall A, B \in H$, $A \cup B \in H$.

$\Rightarrow$) $H$ is a ring $\Rightarrow$ $H$ is a semiring (by exam. 2)

$\Leftarrow$) $H$ is a semiring und $A \cup B \in H$.
Prove that $H$ is a ring. We need to check:
(i) $\forall A, B \in H: A \cup B \in H$;
(ii) $\forall A, B \in H: A \setminus B \in H$.

(i) follows from 2).
(ii) $A \setminus B = \underbrace{\bigcup_{k=1}^{n} C_k}_{H \text{ is a semiring}} = \underbrace{\underbrace{\underbrace{C_1 \cup C_2}_{\in H \text{ by 2)}} \cup \ldots}_{\in H} \cup C_n}_{\in H.} \in H$. ◀

③ Let $X = \mathbb{R}$ and the class $H \subset 2^X$ consists of all subsets with finite number of elements. Show that $H$ is a ring. Is $H$ a $\sigma$-ring? Justify your answer.

▶ $H = \{ A \in \mathbb{R} : A$ has finite number of elements $\}$

Show that $H$ is a ring. Need to check:

1) $\forall A, B \in H : A \cup B \in H$;

2) $\forall A, B \in H : A \setminus B \in H$.

1) Let $A, B \in H$ ($A$ and $B$ have finite number of elem.) Then $A \cup B$ has also finite number of elem. $\Rightarrow$

$\Rightarrow A \cup B \in H$.

2) $A, B \in H$. Then $A \setminus B$ has also finite number of elem.

$\Rightarrow A \setminus B \in H$.

Hence, $H$ is a ring.

But $H$ is not a $\sigma$-ring. Let $A_k = \{x_k\}, k = 1, 2, \dots$, $A_k \in H$, but $\bigcup\limits_{k=1}^{\infty} A_k \notin H$ because $\bigcup\limits_{k=1}^{\infty} A_k$ has countable number of elem.

◀

④ Let $X$ be an uncountable set, and let $H$ be the class of subsets of $X$, which either are countable or have countable complements. Prove or disprove that $H$ is a $\sigma$-algebra.

- Prove that $H$ is a $\sigma$-algebra.

$H = \{A \in X: A$ is a countable or $A^c$ is a countable$\}$

By definition $H$ is a $\sigma$-algebra if and only if:

1) $X \in H$;
2) $A_1, A_2, \dots \in H \Rightarrow \bigcup_{n=1}^{\infty} A_n \in H$;
3) $A \in H \Rightarrow A^c \in H$.

Check 1)-3):

1) $X^c = \emptyset$ and $\emptyset$ is a countable, i.e. $\emptyset \in H$ and $X^c \in H \Rightarrow X \in H$.

2) Assume that:
 a) $A_1, A_2, \dots$ are countable. Then $\bigcup_{n=1}^{\infty} A_n$ is also a countable.
 b) Let $A_k$ is uncountable. Then $A_k^c$ is countable.
 $A_k \subseteq \bigcup_{n=1}^{\infty} A_n \Rightarrow \left(\bigcup_{n=1}^{\infty} A_k\right)^c \subseteq \underbrace{A_k^c}_{\text{countable}} \Rightarrow \left(\bigcup_{n=1}^{\infty} A_n\right)^c$ is also countable and $\bigcup_{n=1}^{\infty} A_n \in H$.

3) Take $A \in H$ and assume that $A$ is a countable.
 $\underbrace{A}_{\text{count.}} = (A^c)^c \Rightarrow$ Complement of $A^c$ is a countable, because $A$ is a countable $\Rightarrow A^c \in H$.

 If the complement of is a countable, then $A^c \in H$.
 Hence, for $A \in H$ we have $A^c \in H$.

$H$ is a $\sigma$-algebra!

(5) Let $H_1, H_2, H \subset 2^X$.

a) Show that the inclusion $H_1 \subset H_2$ implies $\sigma(H_1) \subseteq \sigma(H_2)$.

b) Let $H \subset 2^X$. Show that $\sigma(H) = \sigma(\sigma(H))$.

▶ a) $H_1 \subset H_2 \subseteq \sigma(H_2)$ – $\sigma(H_2)$ is a $\sigma$-algebra which contains $H_1$.

$\sigma(H_1)$ is the smallest $\sigma$-alg. which contains $H_1$.

Then $\sigma(H_1) \subseteq \sigma(H_2)$.

b) $\Rightarrow$) $H \subset \sigma(H) \subseteq \sigma(\sigma(H))$ (*) be defin. of $\sigma$-algebra.

$\Leftarrow$) Prove that $\sigma(\sigma(H)) \subseteq \sigma(H)$ (**)

$\sigma(H)$     $A$ from remark

Remark. If we want to prove that $\sigma(H) \subseteq A$ then it is enough to show that

1) $H \subseteq A$;

2) $A$ is a $\sigma$-algebra.

Check 1) and 2):

1) $\sigma(H) \subseteq \sigma(H)$;

2) $\sigma(H)$ is a $\sigma$-algebra.

Then we have $\sigma(\sigma(H)) \subseteq \sigma(H)$.

(*) and (**) imply $\sigma(H) = \sigma(\sigma(H))$. ◀

⑥ Let $B \subset X$ be fixed and $H \subset 2^X$. Show that $\sigma r(H \cap B) = \sigma r(H) \cap B$. Here $H \cap B = \{A \cap B : A \in H\}$ and $\sigma r(H) \cap B = \{A \cap B : A \in \sigma r(H)\}$.

▶ Let $B \subset X$, $H \subset 2^X$.
We need to show that

$$\sigma r(H \cap B) = \sigma r(H) \cap B,$$

where
$$\sigma r(H) \cap B = \{A \cap B : A \in \sigma r(H)\},$$
$$H \cap B = \{A \cap B : A \in H\}.$$

$\Rightarrow)$ $\sigma r(H \cap B) \subset \sigma r(H) \cap B$ ?

a) $H \cap B \subseteq \sigma r(H) \cap B$ due to $H \subset \sigma r(H)$.

b) $\sigma r(H) \cap B$ is a $\sigma$-ring. Indeed,

- let $A_1, A_2, \ldots \in \sigma r(H) \cap B$. Then there exists $\tilde{A}_1, \tilde{A}_2, \ldots \in \sigma r(H)$ such that
$$A_1 = \tilde{A}_1 \cap B, \quad A_2 = \tilde{A}_2 \cap B, \ldots$$
Then $\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} (\tilde{A}_n \cap B) = \Big(\underbrace{\bigcup_{n=1}^{\infty} \tilde{A}_n}_{\in \sigma r(H)}\Big) \cap B$.

Hence, $\bigcup_{n=1}^{\infty} A_n \in \sigma r(H) \cap B$.

- let $A, C \in \sigma r(H) \cap B$. Then there exists $\tilde{A}, \tilde{C} \in \sigma r(H)$ s.t.
$$A = \tilde{A} \cap B, \quad C = \tilde{C} \cap B.$$
Hence, $A \setminus C = (\tilde{A} \cap B) \setminus (\tilde{C} \cap B) = (\underbrace{\tilde{A} \setminus \tilde{C}}_{\in \sigma r(H)}) \cap B \in \sigma r(H) \cap B$

a), b) $\Rightarrow$ $\sigma r(H \cap B) \subseteq \sigma r(H) \cap B$. !

$\Leftarrow$) $\sigma r(H \cap B) \supseteq \sigma r(H) \cap B$ ?

Take $\mathcal{R} = \{A \in \sigma r(H) : A \cap B \in \sigma r(H \cap B)\}$.

Our goal is to show that

$$\mathcal{R} = \sigma r(H). \ ?$$

Then the defin. of $\mathcal{R}$ will imply that

$$\sigma r(H) \cap B \subseteq \sigma r(H \cap B).$$

because $\forall A \in \sigma r(H)$ $A \cap B \in \sigma r(H \cap B)$.

$\mathcal{R} \subseteq \sigma r(H)$ by the construction. !

Let us show that $\sigma r(H) \subseteq \mathcal{R}$.

1) $H \subseteq \mathcal{R}$ ?

$\forall A \in H$ $A \cap B \in H \cap B \subseteq \sigma r(H \cap B) \Rightarrow A \in \mathcal{R}$.

2) $\mathcal{R}$ is a $\sigma$-ring ?

- Take $A_1, A_2, \ldots \in \mathcal{R}$. Then
  $A_1 \cap B, A_2 \cap B, \ldots \in \sigma r(H \cap B)$
  and
  $\left(\bigcup_{n=1}^{\infty} A_n\right) \cap B = \bigcup_{n=1}^{\infty} (A_n \cap B) \in \sigma r(H \cap B)$.
  Hence $\bigcup_{n=1}^{\infty} A_n \in \mathcal{R}$.

- Take $A, C \in \mathcal{R}$, then
  $A \cap B, C \cap B \in \sigma r(H \cap B)$
  $\Rightarrow (A \setminus C) \cap B = (A \cap B) \setminus (C \cap B) \in \sigma r(H \cap B)$
  Hence $A \setminus C \in \mathcal{R}$.

$\mathcal{R}$ is a $\sigma$-ring and $H \subset \mathcal{R}$.

Consequently $\sigma r(H) \subseteq \mathcal{R}$. !

⑦ Let $X = [0;2]$ and $H = \{\{0\}, [0,1)\}$.
Construct $r(H)$ and $a(H)$.

▸ $r(H) = \{\{0\}, [0,1), (0,1), \emptyset\}$.
$a(H) = \{\{0\}, [0,1), (0,1), \emptyset, X, (0;2], [1,2], \{0\} \cup [1;2]\}$.

◂

⑧ Let $H = \{(a,b] : -\infty < a < b < +\infty\} \cup \{\emptyset\}$.
Show that $\sigma(H) = \mathcal{B}(\mathbb{R})$.

▸ $\mathcal{B}(\mathbb{R}) = \sigma(\{[a,b) : -\infty < a < b < +\infty\} \cup \{\emptyset\})$
We need to show that $\sigma(H) = \mathcal{B}(\mathbb{R})$.

$\Rightarrow$) $\sigma(H) \subseteq \mathcal{B}(\mathbb{R})$ ?
Check two conditions: a) $\{(a,b] : -\infty < a < b < +\infty\} \subseteq \mathcal{B}(\mathbb{R})$;
b) $\mathcal{B}(\mathbb{R})$ - $\sigma$-algebra.

a) $(a,b] = (\underbrace{[a,b)}_{\in \mathcal{B}(\mathbb{R})} \cup \underbrace{\{b\}}_{\in \mathcal{B}(\mathbb{R})}) \setminus \underbrace{\{a\}}_{\in \mathcal{B}(\mathbb{R})} \in \mathcal{B}(\mathbb{R})$ !

b) $\mathcal{B}(\mathbb{R})$ - $\sigma$-alg. by definition. !

$\Leftarrow$) $\sigma(H) \supseteq \mathcal{B}(\mathbb{R})$ ?
Check that: a) $\{[a,b) : -\infty < a < b < +\infty\} \cup \{\emptyset\} \subseteq \sigma(\{(a,b] : -\infty < a < b < +\infty\})$
b) $\sigma(H)$ - $\sigma$-alg.

a) $[a,b) = \bigcap_{n=1}^{\infty} \underbrace{(a - \frac{1}{n}; b)}_{= \bigcup_{k=1}^{\infty} (a - \frac{1}{n}, b - \frac{1}{k}]} = \bigcap_{n=1}^{\infty} \left( \bigcup_{k=1}^{\infty} \underbrace{(a - \frac{1}{n}; b - \frac{1}{k}]}_{\in H} \right) \in \sigma(H)$

b) $\sigma(H)$ - $\sigma$-algebra. !

Hence, $\sigma(H) = \mathcal{B}(\mathbb{R})$.

◂

⑨ Show that there exists a class $H$ consisting of countable number of sets from $\mathbb{R}^2$ such that $\mathcal{B}(\mathbb{R}^2) = \sigma(H)$.

▶ $\mathcal{B}(\mathbb{R}^2) = \sigma\{ [a,b) \times [c,d) : -\infty < a < b < +\infty,\ -\infty < c < d < +\infty \}$

Let $H = \{ [\tilde{a}, \tilde{b}) \times [\tilde{c}, \tilde{d}) : -\infty < \tilde{a} < \tilde{b} < +\infty,\ -\infty < \tilde{c} < \tilde{d} < +\infty,\ \tilde{a}, \tilde{b}, \tilde{c}, \tilde{d} \in \mathbb{Q} \}$

We need to prove that
$$\mathcal{B}(\mathbb{R}^2) = \sigma(H)$$

$\Rightarrow$) $\mathcal{B}(\mathbb{R}^2) \subseteq \sigma(H)$ ?

There exist sequences $a_k \in \mathbb{Q}$ and $b_n \in \mathbb{Q}$, $k, n \in \mathbb{N}$ and such that $a_k \uparrow a$, $b_n \uparrow b$, $a_k < b_n$
$$[a; b) = \bigcap_{k=1}^{\infty} \bigcup_{n=1}^{\infty} [a_k, b_n)$$

Therefore we can write that
$$[a; b) \times [c, d) = \bigcap_{k=1}^{\infty} \bigcup_{n=1}^{\infty} \Big( \underbrace{[a_k, b_n)}_{\in H} \times \underbrace{[c_k, d_n)}_{\in H} \Big) \in \sigma(H),$$

where $a_k, b_n, c_k, d_n \in \mathbb{Q}$, $a_k \uparrow a$, $b_n \uparrow b$, $c_k \uparrow c$, $d_n \uparrow d$.

$\mathcal{B}(\mathbb{R}^2) \subseteq \sigma(H)$, because $\sigma(H)$ is a $\sigma$-algebra.

$\Leftarrow$) $\sigma(H) \subseteq \mathcal{B}(\mathbb{R}^2)$ ?

$H \subseteq \mathcal{B}(\mathbb{R}^2)$ by the definition of $\mathcal{B}(\mathbb{R}^2)$ $\Rightarrow$

$\Rightarrow \sigma(H) \subseteq \mathcal{B}(\mathbb{R}^2)$, since $\mathcal{B}(\mathbb{R}^2)$ is a $\sigma$-alg.

Hence, $\mathcal{B}(\mathbb{R}^2) = \sigma(H)$. ◀

(10) Prove the Theorem 2.18 from Lecture note 2. Namely, let the class $H$ consists of all open sets from $\mathbb{R}$. Show that $\mathcal{B}(\mathbb{R}) = \sigma(H)$.

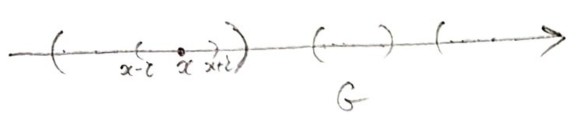
▶ Let $H = \{G \subset \mathbb{R} : G - \text{open}\}$.
We need to show that $\mathcal{B}(\mathbb{R}) = \sigma(H)$.

$\Rightarrow)$ $\mathcal{B}(\mathbb{R}) \subseteq \sigma(H)$ ?

Let $\tilde{H} = \{(a, b) : a < b\}$.
Since $(a, b)$ - open, we can conclude that
$$\tilde{H} \subset H \subset \sigma(H).$$
Since $\sigma(H)$ is a $\sigma$-algebra which consists of all sets from $\tilde{H}$, we get
$$\mathcal{B}(\mathbb{R}) = \sigma(\tilde{H}) \subset \sigma(H). !$$

$\Leftarrow)$ $\mathcal{B}(\mathbb{R}) \supseteq \sigma(H)$ ?

We take an open set $G \subset \mathbb{R}$ and check that $G \in \mathcal{B}(\mathbb{R})$.
For every $x \in G$ we take
$$r_x = \sup\{r \geq 0 : (x - r, x + r) \subset G\}$$




Remark that $r_x > 0$ because $G$ is open.
Next, let us show that $G = \bigcup_{x \in \mathbb{Q} \cap G} (x - \frac{r_x}{2}, x + \frac{r_x}{2})$.

The inclusion $\supset$ is trivial.
We show $\subset$. Let $\tilde{x} \in G$. Then there exists $\tilde{r} > 0$ such that
$$(x - \tilde{r}, x + \tilde{r}) \subset G.$$
Take $x \in \mathbb{Q}$ such that $|\tilde{x} - x| < \frac{r}{4}$.
We want to show that
$$\tilde{x} \in (x - \frac{r_x}{2}, x + \frac{r_x}{2}). \qquad (*)$$

This will imply $\tilde{x} \in \bigcup_{x \in \mathbb{Q} \cap A} (x - \frac{r_x}{2}, x + \frac{r_x}{2})$.

Remark that

$$\left(x - \frac{3\tilde{r}}{4}, x + \frac{3\tilde{r}}{4}\right) \subset (\tilde{x} - \tilde{r}, \tilde{x} + \tilde{r}) \subset G \quad \text{due to}$$

$$\tilde{x} - \tilde{r} = \tilde{x} - x + x - \tilde{r} < \frac{\tilde{r}}{4} + x - \tilde{r} = x - \frac{3\tilde{r}}{4}.$$

$$\tilde{x} + \tilde{r} = \tilde{x} - x + x + \tilde{r} > -\frac{\tilde{r}}{4} + x + \tilde{r} = x + \frac{3\tilde{r}}{4}.$$

This implies that $r_x \geq \frac{3\tilde{r}}{4}$ ($\ast\ast$) because $r_x$ is the supremum of all $r$ such that $(x - r, x + r) \subset G$ and $r = \frac{3\tilde{r}}{4}$ is some $r$ for which $(x - r, x + r) \subset G$.

Let us show that $\tilde{x} \in (x - \frac{r_x}{2}, x + \frac{r_x}{2})$.

By ($\ast$) $\quad x - \frac{3\tilde{r}}{4} \geq \tilde{x} - \tilde{r} \Rightarrow x - \tilde{x} \geq -\frac{\tilde{r}}{4}$.

$\quad x + \frac{3\tilde{r}}{4} \leq \tilde{x} + \tilde{r} \Rightarrow x - \tilde{x} \leq \frac{\tilde{r}}{4}$.

Hence $|x - \tilde{x}| \leq \frac{\tilde{r}}{4} \overset{\text{by } (\ast\ast)}{\leq} \frac{4}{3} \cdot \frac{\tilde{r}}{4} = \frac{\tilde{r}}{3} < \frac{\tilde{r}}{2}$.

We have obtained

$$G = \bigcup_{x \in \mathbb{Q} \cap G} \left(x - \frac{r_x}{2}, x + \frac{r_x}{2}\right)$$

$\Rightarrow \quad G \in \mathcal{B}(\mathbb{R}) \Rightarrow \sigma(H) \subseteq \mathcal{B}(\mathbb{R})$. !

$\blacktriangle$

(11) Let for every $n \geq 1$ a set $A_n$ contains countable number of elements. Show that $\bigcup_{n=1}^{\infty} A_n$ also contains a countable number of elements.

▶ Let $A_1 = \{a_1^1, a_2^1, a_3^1, \dots\}$
$A_2 = \{a_1^2, a_2^2, a_3^2, \dots\}$
$- - - - - - - - -$
$A_n = \{a_1^n, a_2^n, a_3^n, \dots\}$
$- - - - - - - - -$

$A_n$ contains countable number of elements.

$$\begin{array}{lcccc}
A_1: & a_1^1 & a_2^1 & a_3^1 & \dots \\
A_2: & a_1^2 & a_2^2 & a_3^2 & \dots \\
A_3: & a_1^3 & a_2^3 & a_3^3 & \dots \\
A_4: & a_1^4 & a_2^4 & a_3^4 & \dots \\
\dots & \dots & \dots & \dots & \dots
\end{array}$$

This is a way how we can numerate all elements of $\bigcup_{n=1}^{\infty} A_n$.

$\bigcup_{n=1}^{\infty} A_n$ has a countable number of elem. ◀

## 10.1 Solution, proposed by one of students.

Let $G \subseteq \mathbb{R}$ be an open set. Then there exist at most countable number of intervals $(a_k, b_k), k \geq 1$, such that

$$G = \bigcup_k (a_k, b_k).$$

For every $x \in G$ there exists $\varepsilon_x > 0$ such that $(x - \varepsilon_x, x + \varepsilon_x) \subset G$, by the definition of open set.

Next we take rational points $p_x, q_x$ such that

$$x - \varepsilon_x < p_x < x < q_x < x + \varepsilon_x.$$

Consider the set of all ends

it is not interval, just a point in $\mathbb{Q}^2$.

$$\tilde{P} = \{(p_x, q_x) \in \mathbb{Q}^2 : x \in G\}$$

$$\tilde{P} \subseteq \mathbb{Q}^2 \text{ - countable}$$

Hence, $\tilde{P}$ is at most countable.

Consequently, there exists at most countable number of distinct intervals

$$(p_x, q_x), \quad x \in G.$$

Let us numerate them as $(a_k, b_k)$, $k \geq 1$.

Then

$$G = \bigcup_{x \in G} \{x\} = \underbrace{\bigcup_{x \in G} (p_x, q_x)}_{\substack{\text{here are only} \\ \text{at most countable} \\ \text{number of inter-} \\ \text{vals } (a_k, b_k),\, k \geq 1.}} = \bigcup_{k \geq 1} (a_k, b_k).$$