① Let $X=\mathbb{R}$. Consider the following semiring
$$H:=\{(k,k+1]: k\in\mathbb{Z}\}\cup\{\emptyset\}, \text{ - semiring}$$
and define a measure μ on H as follows
$$\mu(\emptyset):=0, \quad \mu((k;k+1]):=1, \quad k\in\mathbb{Z}.$$
Let the measure $\bar{\mu}$ be the extension of μ to the ring $r(H)$ generated by H.

a) Compute $\bar{\mu}((0;1])$, $\bar{\mu}((1;2]\cup(5;6])$ and $\bar{\mu}((-1;3])$.

b) construct the outer measure μ^* generated by $\bar{\mu}$ and compute $\mu^*(\{\frac{1}{2}\})$, $\mu^*((\frac{1}{2};\frac{3}{2}))$ and $\mu^*(\mathbb{N})$.

▶ a) Let $A\in r(H)$, then $A=\bigcup_{k=1}^{n}A_k$, $A_k\cap A_j=\emptyset$, $k\neq j$, $A_k\in H$.

[1p]

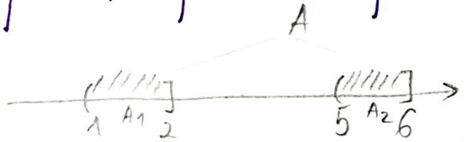
$$\bar{\mu}(A)=\sum_{k=1}^{n}\mu(A_k), \quad A_k\in H.$$

1)

$A=(0;1]\in r(H)$ and $A\in H$ $\quad \left(\underbrace{A}_{\in r(H)}=\bigcup_{k=1}^{1}\underbrace{A_k}_{\in H}\right)$

$\bar{\mu}(A)=\mu(A) \Rightarrow \bar{\mu}((0;1])=\mu((0;1])=1.$

2)

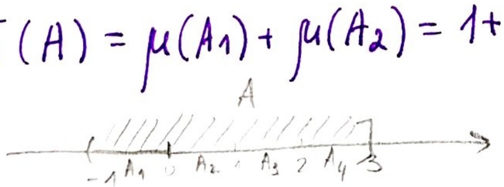


$A=(1;2]\cup(5;6]\in r(H)$, $A_1=(1;2]$, $A_2=(5;6]$, $A_1,A_2\in H$

$A=A_1\cup A_2$, $A_1\cap A_2=\emptyset$

$\bar{\mu}(A)=\mu(A_1)+\mu(A_2)=1+1=2.$

3)



$A=(-1;3]\in r(H)$, $A_1=(-1;0]$, $A_2=(0;1]$, $A_3=(1;2]$, $A_4=(2;3]$,
$A_i\in H$, $i=\overline{1,4}$, $A_i\cap A_j=\emptyset$, $i\neq j$.

$$A=\bigcup_{i=1}^{4}A_i$$

$$\bar{\mu}(A)=\sum_{i=1}^{4}\mu(A_i)=\mu(A_1)+\mu(A_2)+\mu(A_3)+\mu(A_4)=$$
$$=1+1+1+1=4.$$

b) Let $A \subset \mathbb{R}$. By the definition of μ^* we have

$$\mu^*(A) = \begin{cases} \emptyset, & A = \emptyset; \\ \inf \left\{ \sum_{k=1}^{\infty} \mu(A_k), \ A_k \in M, \ A \subset \bigcup_{k=1}^{\infty} A_k \right. . \end{cases}$$

Let A be fixed. We define the set of indixes

$$I_A = \{ k \in \mathbb{Z} : A \cap (k; k+1] \neq \emptyset \}.$$

For instance

- $A = \{\frac{1}{2}\} \Rightarrow I_A = \{0\}$
- $A = (\frac{1}{2}; \frac{3}{2}) \Rightarrow I_A = \{0; 1\}$
- $A = \mathbb{N} \Rightarrow I_A = \mathbb{Z}$.

Let us show that

$$\mu^*(A) = \{\text{the number of elements in } I_A\} =: |I_A|.$$

By the construction of I_A,

$$A \subseteq \bigcup_{k \in I_A} (k; k+1].$$

Hence, $\mu^*(A) \leq \sum_{k \in I_A} \mu(A_k) = \sum_{k \in I_A} 1 = |I_A|$.

On the other hand side, if A_ℓ, $\ell \geq 1$, is a cover of A, that is,
$$A \subseteq \bigcup_{\ell=1}^{\infty} A_\ell, \quad A_\ell \in M,$$
then
$$A \subseteq \bigcup_{k \in I_A} (k; k+1] \subseteq \bigcup_{\ell=1}^{\infty} A_\ell.$$

Indeed, take any $k \in I_A$.
Then $A \cap (k; k+1] \neq \emptyset$, by the construction of I_A.
So, $\exists \, \ell \geq 1$ s.t.
$$A \cap (k; k+1] \subseteq A_\ell \in M.$$
Since all sets of M are intervals of the form $(i; i+1]$, we can conclude that
$$(k; k+1] = A_\ell.$$

So, $(k;k+1] \subseteq \bigcup_{\ell=1}^{\infty} A_\ell \quad , \quad \forall k \in I_A$

Hence $\bigcup_{k \in I_A} (k;k+1] \subseteq \bigcup_{\ell=1}^{\infty} A_\ell$.

This implies that

$$|I_A| = \sum_{k \in I_A} \mu((k;k+1]) \leq \sum_{\ell=1}^{\infty} \mu(A_\ell).$$

So, $|I_A| \leq \inf\left\{\sum_{\ell=1}^{\infty} \mu(A_\ell) : A_\ell \in M,\ A \subseteq \bigcup_{\ell=1}^{\infty} A_\ell\right\} = \mu^*(A)$.

We have proved that

$$\mu^*(A) = |I_A|.$$

Consequently,

$\mu^*(\{\tfrac{1}{2}\}) = |\{0\}| = 1$.

$\mu^*((\tfrac{1}{2};\tfrac{3}{2})) = |\{0;1\}| = 2$.

$\mu^*(\mathbb{N}) = |\mathbb{Z}| = +\infty$.

② Let λ^* be an outer measure on 2^X. Show that
[2p] a set $A \in 2^X$ is λ^* measurable if and only if
$\forall\, U \subset A$ and $\forall\, V \subset A^c$ $\lambda(U \cup V) = \lambda^*(U) + \lambda^*(V)$.

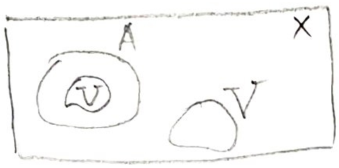
▶ A set A is λ^*-measurable if

$$\forall B \subseteq X \quad \lambda^*(B) = \lambda^*(B \cap A) + \lambda^*(B \setminus A). \qquad (1)$$

We need to show that a set A is λ^*-meas. if and only if

$$\forall\, U \subset A \text{ and } \forall\, V \subset A^c \quad \lambda^*(U \cup V) = \lambda^*(U) + \lambda^*(V). \quad (2)$$

$\Rightarrow$) Let we have (1). Prove (2).



Take $U \subset A$ and $V \subset A^c$.

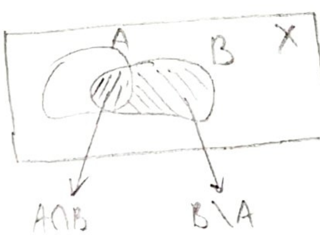
Let $B := U \cup V \subseteq X$

$$\lambda^*(\underbrace{U \cup V}_{B}) = \lambda^*(B) \overset{(1)}{=} \lambda^*(\underbrace{B \cap A}_{U}) + \lambda^*(\underbrace{B \setminus A}_{V}) =$$

$$= \lambda^*(U) + \lambda^*(V) \ !$$

Hence, $\lambda^*(U \cup V) = \lambda^*(U) + \lambda^*(V)$.

$\Leftarrow$) We have (2) and we need to prove (1).



Take $B \subseteq X$. Let $U = A \cap B \subset A$,
$V = B \setminus A \subset A^c$.

We have by (2):

$$\lambda^*(B) := \lambda^*(\underbrace{U \cup V}_{= B}) = \lambda^*(\underset{A \cap B}{U}) + \lambda^*(\underset{B \setminus A}{V}) = \lambda^*(A \cap B) + \lambda^*(B \setminus A)$$

$$\Rightarrow \quad \lambda^*(B) = \lambda^*(A \cap B) + \lambda^*(B \setminus A) \Rightarrow A \text{ is } \lambda^*\text{-measurable}$$

◀

③ Let μ^* be the outer measure generated by a measure μ defined on a ring R, and let S denotes the set of all μ^*-measurable sets.
[2p] Show that $\sigma r(R) \subset \sigma(R) \subset S$.

▶ We need to show that

1) $\sigma r(R) \subset \sigma(R)$ and 2) $\sigma(R) \subset S$.

1) $\sigma r(R)$ is the smallest σ-ring, which contains all sets of R by the definition of the σ-ring generated by R.
A σ-algebra $\sigma(R)$ is a σ-ring, which contains all sets of R.
By the definition of the σ-ring generated by R we have that

$$\sigma r(R) \subset \sigma(R).$$

2) Show that $\sigma(R) \subset S$. For this check that

a) $R \subset S$
b) S is a σ-algebra.

a) by Theorem 4.12 $R \subset S$.
b) S is a class of all μ^*-measurable sets and μ^* is the outer measure generated by a meas. μ defined on a ring R.
Then by Caratheodory theorem S is a σ-algebra.

Hence, by 1) and 2)

$$\sigma r(R) \subset \sigma(R) \subset S.$$

④ Let $X = \mathbb{R}$ and λ be the Lebesgue measure. Denote by S the class of all Lebesgue measurable subsets of $\mathbb{R}$.

a) Let $A \in S$, $\lambda(A) < +\infty$ and $f(x) := \lambda(A \cap (-\infty; x))$, $x \in \mathbb{R}$. Show that the function f is countinuous on $\mathbb{R}$.

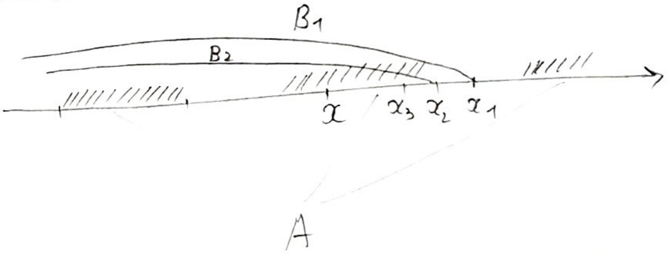
b) Let A be a bounded set and $\lambda(A) > 0$. Prove that there for every $\alpha \in (0; \lambda(A))$ there exists $B \subset A$, $B \in S$ such that $\lambda(B) = \alpha$.

▶ a) We need to show that f is contineuous on $\mathbb{R}$. For this prove that f is left and right contineous at x for $\forall x \in \mathbb{R}$.

I. Let the sequence $x_n \searrow x$ and show that

$$\lim_{n\to\infty} f(x_n) = f(x).$$

$$f(x_n) = \lambda(\underbrace{A \cap (-\infty; x_n)}_{B_n}), \quad f(x) = \lambda(A \cap (-\infty; x))$$



Let $B_n = A \cap (-\infty; x_n)$.

Then $B_1 \subset A$ and

$$\lambda(B_1) \le \lambda(A) < +\infty \text{ (given)}$$

moreoven, the sequence B_n decreases. ($B_{n+1} \subset B_n$).

Use the continuity of measure from above

$$\bigcap_{n=1}^{\infty} B_n = A \cap (-\infty; x] = (A \cap (-\infty; x)) \cup (A \cap \{x\})$$

$$\lambda\left(\bigcap_{n=1}^{\infty} B_n\right) \overset{\text{cont. from above}}{=} \lim_{n\to\infty} \lambda(B_n) = \lim_{n\to\infty} \underbrace{\lambda(A \cap (-\infty; x_n))}_{f(x_n)} = \lim_{n\to\infty} f(x_n)$$

On the other hand side

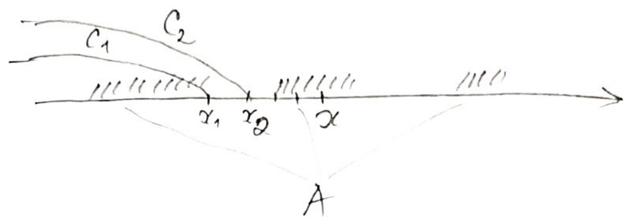
$$\lambda\left(\bigcap_{n=1}^{\infty} B_n\right) = \underbrace{\lambda(A \cap (-\infty; x))}_{f(x)} + \underbrace{\lambda(A \cap \{x\})}_{=0} = f(x)$$

because $A \cap \{x\} \subseteq \{x\}$ and $\lambda(A \cap \{x\}) \le \lambda(\{x\}) = 0$

Hence, $\lim_{n\to\infty} f(x_n) = f(x)$ if $x_n \searrow x$.

II. Show that f is left continuous at x.
Take the sequence $x_n \nearrow x$ and show that
$$\lim_{n\to\infty} f(x_n) = f(x).$$

$$f(x_n) = \lambda(\underbrace{A \cap (-\infty; x_n)}_{C_n}), \quad f(x) = \lambda(A \cap (-\infty; x)).$$



Let $C_n = A \cap (-\infty; x_n)$,

$C_n \subseteq C_{n+1}$ and

$$\bigcup_{n=1}^{\infty} C_n = A \cap (-\infty; x), \quad \forall n \geq 1.$$

$$\lambda\left(\bigcup_{n=1}^{\infty} C_n\right) \overset{\text{cont. from below}}{=} \lim_{n\to\infty} \lambda(C_n) = \lim_{n\to\infty} \underbrace{\lambda(A \cap (-\infty; x_n))}_{f(x_n)} = \lim_{n\to\infty} f(x_n).$$

On the other hand side

$$\lambda\left(\bigcup_{n=1}^{\infty} C_n\right) = \lambda(A \cap (-\infty; x)) = f(x).$$

Hence, $\lim_{n\to\infty} f(x_n) = f(x)$ if $x_n \nearrow x$.

By I and II we have that f is continuous on $\mathbb{R}$.

(4) b) We have that A is a bounded set and $\lambda(A)>0$.

[3 p.] We need to show that

$$\exists B \subset A,\ B \in S \text{ s.t. } \lambda(B)=\alpha,\ \alpha \in (0; \lambda(A)).$$

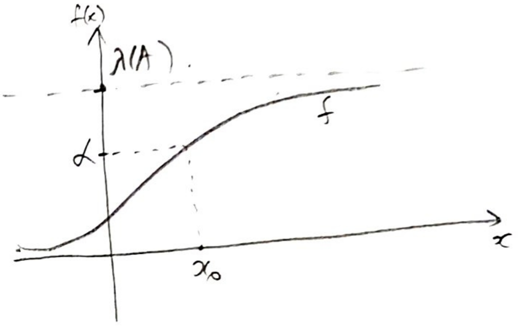
Let $f(x)=\lambda(A\cap(-\infty;x))$. Then

$$f(-\infty) \overset{\text{by a)}}{=} \lim_{x\to-\infty} f(x) = 0,$$

$$f(+\infty) = \lim_{x\to+\infty} f(x) = \lambda(A).$$

f is a continuous on $\mathbb{R}$ and α is a value between 0 and $\lambda(A)$. Then $\exists x_0 \in \mathbb{R}$ s.t. $f(x_0)=\alpha$

$$f(x_0) = \lambda(\underbrace{A\cap(-\infty;x_0)}_{B}).$$



Let $B = A\cap(-\infty;x_0)$,

$\underline{B\subset A}$, $\underline{B\in S}$.

and $\underline{\lambda(B)} = f(x_0) = \underline{\alpha}$.

(5) Let $X = \mathbb{R}^2$ and λ be the Lebesgue measure. Denote by S the class of all Lebesgue measurable subsets of $\mathbb{R}^2$. Show that

[1p] a) a one-point set $\{(x,y)\}$ belongs to S and $\lambda(\{(x,y)\}) = 0$ for every $(x,y) \in \mathbb{R}^2$;

[2p] b) the interval $I = \{(x,y) : x \in [a,b], y = 1\}$ belongs to S and $\lambda(I) = 0$ for every $a < b$;

[2p] c) the line $L = \{(x,y) : x \in \mathbb{R}, y = 1\}$ belongs to S and $\lambda(L) = 0$;

[3p] d) the set $F = \{(x,y) : 0 \le x \le 1, 0 \le y \le f(x)\}$ belongs to S and $\lambda(F) = \int_0^1 f(x)\,dx$, where f is a nonnegative continuous function on $[0,1]$.

▸ a) $X = \mathbb{R}^2$, $H = \{[a_1, b_1) \times [a_2, b_2),\ -\infty < a_i < b_i < +\infty\} \cup \{\emptyset\}$, $i = 1,2$.

H is a semiring.

$\lambda(A) = \lambda([a_1, b_1) \times [a_2, b_2)) = (b_1 - a_1)\cdot(b_2 - a_2)$, $A \in H$. $\Rightarrow$ λ is a measure on H.

$\lambda(\emptyset) = 0$

We know that $\sigma(H) = \mathcal{B}(\mathbb{R}^2) \subset S$.

Let $A = \{(x,y)\}$, $A_n = [x, x+\frac{1}{n}) \times [y; y + \frac{1}{n}) \in \mathcal{B}(\mathbb{R}^2)$

$$A = \bigcap_{n=1}^{\infty} \underbrace{A_n}_{\in \mathcal{B}(\mathbb{R}^2)} \in \mathcal{B}(\mathbb{R}^2) \Rightarrow \underline{A \subset S}$$

$A_n \downarrow$ and $\lambda(A_1) = \lambda([x, x+1) \times [y, y+1)) = 1 \cdot 1 = 1 < +\infty$

Then

$$\underline{\lambda(A) = \lambda(\{x,y\})} = \lambda\left(\bigcap_{n=1}^{\infty} A_n\right) \overset{\text{contin from above}}{=} \lim_{n\to\infty} \lambda(A_n) = \frac{1}{n} \cdot \frac{1}{n} = \underline{0}.$$

◂

▸ b) Let $B_n = [a, b + \frac{1}{n}) \times [1, 1 + \frac{1}{n}) \in \mathcal{B}(\mathbb{R}^2)$

$I = \{(x,y) : x \in [a,b], y = 1\}$

$$I = \bigcap_{n=1}^{\infty} \underbrace{B_n}_{\in \mathcal{B}(\mathbb{R}^2)} \in \mathcal{B}(\mathbb{R}^2) \Rightarrow \underline{I \subset S}$$

$B_n \downarrow$ and $\lambda(B_1) = \lambda([a, b+1) \times [1, 2)) = (b + 1 - a)\cdot(2 - 1) = (b + 1 - a) < +\infty$

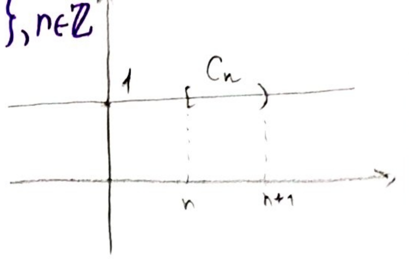
Then

$$\underline{\lambda(I)} = \lambda\left(\bigcap_{n=1}^{\infty} B_n\right) = \lim_{n\to\infty} \lambda(B_n) = \lim_{n\to\infty} (b + \tfrac{1}{n} - a)\cdot(1 + \tfrac{1}{n} - 1) = \underline{0}.$$

◂

c) Let $C_n = \{(x,y) : x \in [n; n+1], y = 1\}, n\in\mathbb{Z}$

$L = \{(x,y) : x \in \mathbb{R}, y = 1\}$



$$L = \bigcup_{n=1}^{\infty} \underbrace{C_n}_{\in B(\mathbb{R}^2) \text{ by b)}} \in B(\mathbb{R}^2) \Rightarrow \underline{\underline{L \subset S}}$$

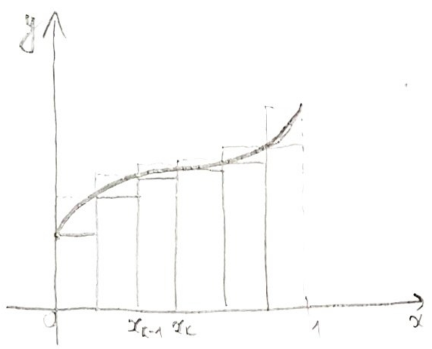
$$0 \le \lambda(L) \underset{\sigma\text{-add.}}{\le} \sum_{n=-\infty}^{\infty} \underbrace{\lambda(C_n)}_{=0 \text{ by b)}} = 0 \Rightarrow \underline{\underline{\lambda(L) = 0}} \; !$$

d) ~~Let~~ $F = \{(x,y) : 0 \le x \le 1, 0 \le y \le f(x)\}$.

We need to show that $\overline{F \subset S}$ and $\lambda(F) = \int_0^1 f(x)dx$.

Take the partition $x_n = \frac{1}{n}$, $k = \overline{1,n}$ and let

$$M_k = \sup_{x\in[x_{k-1};x_k]} f(x), \quad m_k = \inf_{x\in[x_{k-1};x_k]} f(x).$$



Then $U_n = \sum_{k=1}^{n} \overbrace{\frac{1}{n}}^{(x_k - x_{k-1})} \cdot M_k$ - upper Darboux sum of f

$L_n = \sum_{k=1}^{n} \frac{1}{n} \cdot m_k$ - lower Darboux sum of f.

$$A_n^k = \{(x,y) : \frac{k-1}{n} \le x < \frac{k}{n}, 0 \le y < m_k\}$$

$$B_n^k = \{(x,y) : \frac{k-1}{n} \le x < \frac{k}{n}, 0 \le y < M_k\}$$

$$A_n = \bigcup_{k=1}^{n} \underbrace{A_n^k}_{\in B(\mathbb{R}^2)} \in B(\mathbb{R}^2)$$

$$B_n = \bigcup_{k=1}^{n} B_n^k \in B(\mathbb{R}^2)$$

$\lambda(A_n) = L_n$, $\lambda(B_n) = U_n$.

$$A_n \subset F \subset B_n \Rightarrow F \subset S$$

and

$$\lambda(A_n) \le \lambda(F) \le \lambda(B_n)$$

$$\begin{array}{ccccc} \parallel & & & & \parallel \\ L_n & \le & \lambda(F) & \le & U_n \end{array}$$

$$\int_0^1 f(x)dx$$

Hence, $\lambda(F) = \int_0^1 f(x)dx$.

⑥ Let $(X, \mathcal{F})$ and $(X', \mathcal{F}')$ be measurable spaces. Which functions $f: X \to X'$ are $(\mathcal{F}, \mathcal{F}')$-measurable if

a) $\mathcal{F}' = \{\varnothing, X'\}$;

b) $X = [0;1]$, $\mathcal{F} = \sigma(\{[0;\frac{1}{2}]\})$ and $X' = \mathbb{R}$, $\mathcal{F}' = \mathcal{B}(\mathbb{R})$.

▸ Recall that f is $(\mathcal{F}, \mathcal{F}')$-measurable if

$$\forall A' \in \mathcal{F}' \quad f^{-1}(A') = \{x \in X : f(x) \in A'\} \in \mathcal{F}.$$

a) $\mathcal{F}' = \{\varnothing, X'\}$. Then

$$f^{-1}(\varnothing) = \{x \in X : f(x) = \varnothing\} = \varnothing \in \mathcal{F}$$

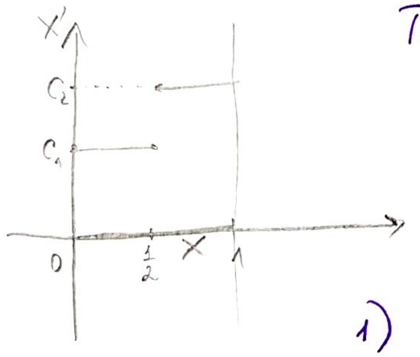
$$f^{-1}(X') = \{x \in X : f(x) = X'\} = X \in \mathcal{F}$$

because $\mathcal{F}$ is a σ-algebra

Hence, any functions f are $(\mathcal{F}, \mathcal{F}')$-measurable.

b) $X = [0;1]$, $\mathcal{F} = \sigma(\{[0;\frac{1}{2}]\}) = \{[0;\frac{1}{2}], (\frac{1}{2};1], X, \varnothing\}$

$X' = \mathbb{R}$, $\mathcal{F}' = \mathcal{B}(\mathbb{R})$.



Take
$$f(x) = \begin{cases} C_1, & x \in [0;\frac{1}{2}] \\ C_2, & x \in (\frac{1}{2};1], \end{cases} \quad C_1, C_2 = \text{const}.$$

For any set $A' \in \mathcal{F}'$ we have:

1) if $C_1, C_2 \notin A'$ then
$$f^{-1}(A') = \{x \in X : f(x) \in A'\} = \varnothing \in \mathcal{F};$$

2) if $C_1 \in A'$ then
$$f^{-1}(A') = \underbrace{\{x \in X : f(x) \in A'\}}_{= \{x \in X : f(x) = C_1\}} = [0;\tfrac{1}{2}] \in \mathcal{F};$$

3) if $C_2 \in A'$ then
$$f^{-1}(A') = \{x \in X : f(x) \in A'\} = \{x \in X : f(x) = C_2\} = (\tfrac{1}{2};1] \in \mathcal{F};$$

4) if $C_1, C_2 \in A'$ then
$$f^{-1}(A') = \{x \in X : f(x) \in A'\} = \{x \in X : f(x) = C_1 \text{ and } f(x) = C_2\} = X = [0;1] \in \mathcal{F}.$$

Hence, f is a $(\mathcal{F}, \mathcal{F}')$-measurable.

7 Define the class of all μ^*-measurable sets, where μ^* is the outer measure from Exercise 1.

▶ $H = \{ (k; k+1] : k \in \mathbb{Z} \} \cup \{\emptyset\}$

$\mu((k; k+1]) = 1$, $k \in \mathbb{Z}$

Let S be a set of all μ^*-measurable sets. By the Caratheodory theorem, we know that S is a σ-algebra and the outer measure μ^* generated by the measure μ on H is a measure on S.

Th. 4.12 implies that
$$H \subseteq S.$$

Hence $\sigma(H) \subseteq S$.

In particular, we can conclude that any set A expressed as a countable union of sets from H is μ^*-measurable, that is,
$$A = \bigcup_{k=1}^{\infty} A_k, \qquad (*)$$
where $A_k \in H$ is μ^*-measurable.

Let us show that only such sets are μ^*-measurable. Assume that A can not be written in the form $(*)$ and $A \in S$. Then exists $C = (k_0, k_0+1] \in H$ s.t.
$$\hat{A} := A \cap C \neq C \text{ and } \hat{A} \neq \emptyset.$$

Since $C \in S$, we have that $\hat{A} \in S$.

Let us show that $\hat{A}$ is not μ^*-measurable.

Take
$$B := C = (k_0; k_0+1].$$

Then
$$\mu^*(B) = 1$$

and
$$\mu^*(B \cap \tilde{A}) + \mu^*(B \setminus \tilde{A}) = \mu^*(A \cap C) + \mu^*(C \setminus A) = 1 + 1 = 2.$$

Since, $A \cap C, A \setminus A \notin \{ (k; k+1], \emptyset \}$.

We got a contradiction. ◀

(8) Fine an example of an outer measure λ^* on 2^X such that the class of all λ^*-measurable sets S equals $\{\emptyset, X\}$

- A set A is λ^*-measurable if

$\forall B \in X \quad \lambda^*(B) = \lambda^*(B \setminus A) + \lambda^*(B \cap A)$,

λ^* – outer measure on 2^X.

1. Take

$$\lambda^*(A) = \begin{cases} 1, & A \neq \emptyset \\ 0, & A = \emptyset \end{cases}, \quad \forall A \in 2^X. \qquad (*)$$

2. Show that λ^* is an outer measure.

By defin. $\lambda^* : 2^X \to (\infty; +\infty]$ is an outer measure if λ^* is nonnegative;

(i) $\lambda^*(\emptyset) = 0$, λ^* is nonnegative;

(ii) $\forall A, A_n \in 2^X$, $A \subset \bigcup_{n=1}^{\infty} A_n$

$$\lambda^*(A) \leq \sum_{n=1}^{\infty} \lambda^*(A_n).$$

Check (i) and (ii): (i) given by $(*)$.

(ii) if $A = \emptyset$ then $\lambda^*(A) = 0$ and $0 \leq \sum_{n=1}^{\infty} \lambda^*(A_n)$, $\forall A_n$ s.t. $A \subset \bigcup_{n=1}^{\infty} A_n$

if $A \neq \emptyset$ then $\exists n_0 : A_{n_0} \neq \emptyset$ and $A \subset \bigcup_{n=1}^{\infty} A_n$, otherwise $\bigcup_{n=1}^{\infty} A_n = \emptyset$

$$\lambda^*(A) \leq \sum_{n=1}^{\infty} \lambda^*(A_n) = \sum_{n=1}^{\infty} 1 \, !$$

Hence, λ^* – is an outer measure.

3. Show that $\emptyset$ and X are λ^*-measurable.

$\forall B \in 2^X \quad \lambda^*(B) = \lambda^*(\underbrace{B \cap \emptyset}_{=\emptyset}) + \lambda^*(\underbrace{B \setminus \emptyset}_{=B}) = \lambda^*(B)$

$\lambda^*(B) = \lambda^*(\underbrace{B \cap X}_{=B}) + \lambda^*(\underbrace{B \setminus X}_{=\emptyset}) = \lambda^*(B)$

Hence, $\emptyset$ and X are λ^*-measurable

4. Show that if $A \neq \emptyset$, $A \neq X$ then A is not λ^*-measurable.

Take $B = X$

$$\lambda^*(B) = \lambda^*(A \cap B) + \lambda^*(B \setminus A)$$

$$\underbrace{\lambda^*(X)}_{\substack{\shortparallel \\ 1}} = \lambda^*(\underbrace{A \cap X}_{=A}) + \lambda^*(\underbrace{X \setminus A}_{A^c}) = \lambda^*(A) + \lambda^*(A^c) = 1+1$$

$1 \neq 2$.

Hence, for $A \neq \emptyset$ and $A \neq X$ a set A is not λ^*-meas.

$S = \{\emptyset, X\}$ - the class of all λ^*-measurable sets, where λ^* given in $(*)$.