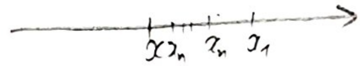
(1) Let $X=\mathbb{R}$, $\mathcal{F}=\mathcal{B}(\mathbb{R})$ and λ be the Lebesgue
[3p] measure on $\mathbb{R}$. Let also $f\in \mathcal{L}(\mathbb{R},\lambda)$.
Show that the function

$$\varphi(x):=\int_{(-\infty;x]} f(t)\,\lambda(dt),\quad x\in\mathbb{R}$$

is continuous on $\mathbb{R}$.

▶ We need to check that
$\forall x\in\mathbb{R}\ \ \forall x_n\to x\quad \varphi(x_n)\to\varphi(x)$.

Take the sequence $x_n\downarrow x$, $n\to\infty$ and show that $|\varphi(x_n)-\varphi(x)|\to 0$ if $n\to\infty$.



$$|\varphi(x_n)-\varphi(x)|=\Big|\int_{(-\infty;x_n]} f\,d\lambda-\int_{(-\infty;x]} f\,d\lambda\Big|=\Big|\int_{(x;x_n]} f\,d\lambda\Big|\le$$

$$\le \underbrace{\int_{(x;x_n]} |f|\,d\lambda}_{=\mu(A_n)}.$$

Let $A_n=(x;x_n]$. The function $|f|\ge 0$, then

$$\mu(A_n)=\int_{(x;x_n]} |f|\,d\lambda$$

and $\mu(A_n)$ is a measure on $\mathcal{F}$.
Show that $\mu(A_n)\to 0$, $n\to\infty$.
Then sequence A_n is decreasing and $\mu(A_1)=\mu((x;x_1])=$
$=\int_{(x;x_1]} |f|\,d\lambda<+\infty$, because f is integrable.

$$\lim_{n\to\infty}\mu(A_n)=\lim_{n\to\infty}\mu((x;x_n])\overset{\text{cont. from above}}{=}\mu\Big(\bigcap_{n=1}^{\infty}A_n\Big)=\mu(\emptyset)=0.$$

Hence, $\mu(A_n)\to 0$, $n\to\infty$. Then $|\varphi(x_n)-\varphi(x)|\to 0$, $n\to\infty$, if $x_n\downarrow x$.

Similarly, take a sequence $x_n\uparrow x$. Show that

$$|\varphi(x_n)-\varphi(x)|\to 0,\ n\to\infty.$$

$$|\varphi(x_n)-\varphi(x)|=\Big|\int_{(x_n;x]} f\,d\lambda\Big|\le\int_{(x_n;x]} |f|\,d\lambda=\mu(B_n),\quad B_n=(x_n;x].$$

$\mu(B_n)\to 0$, because $B_{n+1}\subset B_n$ and

$$\lim_{n\to\infty}\mu(B_n)=\lim_{n\to\infty}\mu((x_n;x])=\mu\Big(\bigcap_{n=1}^{\infty}B_n\Big)=\mu(\{x\})=0.$$

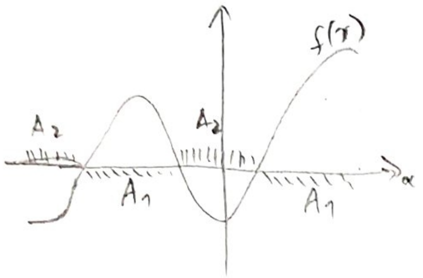
Hence, $\forall x\in\mathbb{R}\ \forall x_n\to x\quad \varphi(x_n)\to\varphi(x)\Rightarrow\varphi(x)$ is continuous on $\mathbb{R}$. ◀

(2) Let $f \in L(X, \lambda)$ and $\int_A f \, d\lambda = 0$ for all $A \in \mathcal{F}$.

[3p] Show that $f = 0$ λ-a.e.

▸ We need to show that $f = 0$ λ-a.e. on X.

We know that if $f \in L(A, \lambda)$, $f \geq 0$ and $\int_A f \, d\lambda = 0$, (*)
then $f = 0$ λ-a.e. on A.



Take the set

$$A_1 = \{x \in X : f(x) \geq 0\}$$

Then

$$\int_{A_1} \underbrace{f}_{\geq 0 \text{ on } A_1} d\lambda = 0 \overset{\text{by } (*)}{\Rightarrow} f = 0 \ \lambda\text{-a.e. on } A_1.$$

Take the set

$$A_2 = \{x \in X : f(x) \leq 0\}.$$

Then

$$-\int_{A_2} f \, d\lambda = \int_{A_2} \underbrace{(-f)}_{\geq 0 \text{ on } A_2} d\lambda = 0 \overset{\text{by } (*)}{\Rightarrow} -f = 0 \ \lambda\text{-a.e on } A_2 \Rightarrow$$

$$\Rightarrow f = 0 \ \lambda\text{-a.e. on } A_2.$$

Since $A_1 \cup A_2 = X$ and $f = 0$ λ-a.e. on A_1 and A_2,
then $f = 0$ λ-a.e. on X.

(3) Let $f_n \to f$ λ-a.e and $f_n \to g$ λ-a.e.
[2p] Show that $f = g$ λ-a.e.

$f_n \to f$ λ-a.e. $\overset{\text{by defin.}}{\Rightarrow}$ $\exists\, \varphi_1 \in \mathcal{F}$ s.t.
$\lim_{n\to\infty} f_n(x) = f(x)$ $\forall x \in X \setminus \varphi_1$ and $\lambda(\varphi_1) = 0$.

$f_n \to g$ λ-a.e. $\Rightarrow$ $\exists\, \varphi_2 \in \mathcal{F}$ s.t.
$\lim_{n\to\infty} f_n(x) = g(x)$ $\forall x \in X \setminus \varphi_2$ and $\lambda(\varphi_2) = 0$.

Take $\varphi = \varphi_1 \cup \varphi_2$. Then $0 \le \lambda(\varphi) \le \lambda(\varphi_1) + \lambda(\varphi_2) = 0 \Rightarrow \lambda(\varphi) = 0$

$\begin{cases} f_n \to f \ \lambda\text{-a.e} \Rightarrow \lim_{n\to\infty} f_n(x) = f(x) \quad \forall x \in X \setminus \varphi. \\ f_n \to g \ \lambda\text{-a.e} \Rightarrow \lim_{n\to\infty} f_n(x) = g(x) \quad \forall x \in X \setminus \varphi \end{cases}$

By the uniqueness of limit for sequences in $\mathbb{R}$
$f(x) = g(x)$ $\forall x \in X \setminus \varphi$ $\Rightarrow$

$\Rightarrow f = g$ λ-a.e.

④ Assume that $f: X \to \mathbb{R}$ satisfies the following property: for every $a > 0$

$$\lambda(\{x \in X : |f(x)| \geq a\} = 0$$

Show that $f = 0$ λ-a.e.

▶ By definition 8.1 $f = 0$ λ-a.e on X if

$$\lambda(\{x \in X : f(x) \neq 0\} = 0.$$

We need to show that $\lambda(\{x \in X : f(x) \neq 0\}) = 0$.

For this prove the equality

$$\underbrace{\{x \in X : f(x) \neq 0\}}_{\overset{\shortparallel}{A}} = \bigcup_{n=1}^{\infty} \underbrace{\{x \in X : |f(x)| \geq \tfrac{1}{n}\}}_{= A_n}.$$

We know that $x \in A \Leftrightarrow \exists\, n$ such that $|f(x)| \geq \frac{1}{n} \Leftrightarrow$

$\Leftrightarrow \exists\, n : x \in A_n \Leftrightarrow x \in \bigcup_{n=1}^{\infty} A_n$.

Hence, $\{x \in X : f(x) \neq 0\} = \bigcup_{n=1}^{\infty} \{x \in X : |f(x)| \geq \frac{1}{n}\}$.

$$\lambda(\{x \in X : f(x) \neq 0\}) = \lambda\left(\bigcup_{n=1}^{\infty} \{x \in X : |f(x)| \geq \tfrac{1}{n}\}\right) \leq$$

$$\leq \sum_{n=1}^{\infty} \underbrace{\lambda(\{x \in X : |f(x)| \geq \tfrac{1}{n}\})}_{= 0 \text{ given}} = 0 \quad \Rightarrow$$

$$\Rightarrow \lambda(\{x \in X : f(x) \neq 0\}) = 0 \overset{\text{by def.}}{\Rightarrow} f = 0 \ \lambda\text{-a.e.}$$

◀

(5) Let $f_n \xrightarrow{\lambda} f$.

a) Show that $|f_n| \xrightarrow{\lambda} |f|$.

b) Let additionally $\lambda(X) < +\infty$. and $g : \mathbb{R} \to \mathbb{R}$ be a continuous function. Prove that
$$g(f_n) \xrightarrow{\lambda} g(f).$$

▶ a) $|f_n| \xrightarrow{\lambda} |f|$ if
$$\forall \varepsilon > 0 \quad \lambda(\{x \in X : ||f_n(x)| - |f(x)|| \geq \varepsilon\}) \to 0, \ n \to \infty.$$

Since
$$||f_n(x)| - |f(x)|| \leq |f_n(x) - f(x)|, \qquad \{ \ ||x| - |y|| \leq |x - y|$$

then
$$\{x \in X : ||f_n(x)| - |f(x)|| \geq \varepsilon\} \subseteq \{x \in X : |f_n(x) - f(x)| \geq \varepsilon\}$$

and
$$\lambda(\{x \in X : ||f_n(x)| - |f(x)|| \geq \varepsilon\}) \leq \underbrace{\lambda(\{x \in X : |f_n(x) - f(x)| \geq \varepsilon\})}_{\to 0,\ n \to \infty,\ \text{because } f_n \xrightarrow{\lambda} f.}$$

Hence, $|f_n| \xrightarrow{\lambda} |f|$.

b) We need to show that $g(f_n) \xrightarrow{\lambda} g(f)$.

$g(f_n) \xrightarrow{\lambda} g(f) \iff$ for every subsequence $\{g(f_{n_k})\}_{k \geq 1}$ exists a subsubsequence $\{g(f_{n_{k_j}})\}_{j \geq 1}$ s.t.
$$g(f_{n_{k_j}}) \to g(f) \quad \lambda\text{-a.e.}$$

Take a $\{g(f_{n_k})\}_{k \geq 1}$ and consider the subsequence $\{f_{n_k}\}$.

Since $f_n \xrightarrow{\lambda} f$ then exists a subsubseq. $\{f_{n_{k_j}}\}_{j \geq 1}$ s.t.
$$f_{n_{k_j}} \to f \quad \lambda\text{-a.e.}$$

The function g is a continuous. Then
$$g(f_{n_{k_j}}) \xrightarrow{\lambda} g(f).$$

◀

6. Let $f_n, n \geq 1$, be non-negative functions.

a) Show that for every $\varepsilon > 0$
$$\varepsilon \lambda(\{x \in X : f_n(x) \geq \varepsilon\}) \leq \int_X f_n \, d\lambda.$$

b) Check that the convergence $\int_X f_n \, d\lambda \to 0$ implies $f_n \xrightarrow{\lambda} 0$.

a) Let $A_n = \{x \in X : f_n(x) \geq \varepsilon\}$ and
[3p]
$$g_n(x) = \varepsilon \cdot \mathbb{I}_{A_n}(x) = \begin{cases} \varepsilon, & x \in A_n \\ 0, & x \notin A_n. \end{cases} \qquad (*)$$

$$\int_X g_n \, d\lambda = \int_{A_n} g_n \, d\lambda + \int_{X \setminus A_n} \underbrace{g_n}_{=0} \, d\lambda = \int_{A_n} \underbrace{g_n}_{=\varepsilon} \, d\lambda = \varepsilon \cdot \lambda(A_n). \qquad (**)$$

For $\forall x \in X$ $g_n(x) \leq f_n(x)$, because

1) if $x \in A_n$, then $g_n(x) \overset{(*)}{=} \varepsilon$ and $f_n(x) \geq \varepsilon$ on A_n $\Rightarrow$
$\Rightarrow g_n(x) \leq f_n(x)$;

2) if $x \notin A_n$, then $g_n(x) \overset{(*)}{=} 0$ and $\underbrace{f_n(x) \geq 0}_{\text{given}}$ $\Rightarrow$
$\Rightarrow g_n(x) \leq f_n(x)$.

From $(**)$ we have that
$$\underbrace{\varepsilon \lambda(A_n)}_{= \varepsilon \lambda(\{x \in X : f(x) \geq \varepsilon\})} = \int_X g_n \, d\lambda \leq \int_X f_n \, d\lambda!$$

b) We need to show that $f_n \xrightarrow{\lambda} 0$. For this check that
[3p]
$$\forall \varepsilon > 0 \quad \lambda(\{x \in X : |f_n(x) - 0| \geq \varepsilon\}) \to 0.$$

Consider $\forall \varepsilon > 0$. Then
$$\lambda(\{x \in X : |f_n(x) - 0| \geq \varepsilon\}) \overset{f_n \geq 0}{=} \lambda(\{x \in X : f_n(x) \geq \varepsilon\}) \overset{\text{by a)}}{\leq}$$
$$\leq \frac{1}{\varepsilon} \int_X f_n \, d\lambda \xrightarrow[\text{given}]{} 0.$$

Hence, $\forall \varepsilon > 0$ $\lambda(\{x \in X : |f_n(x) - 0| \geq \varepsilon\}) \to 0 \Rightarrow f_n \xrightarrow{\lambda} 0.$

(7) Let $\lambda(X) < +\infty$.

a) Prove that $f_n \to f$ λ-a.e. if and only if

$$\forall \varepsilon > 0 \quad \lambda\Big(\bigcup_{k=n}^{\infty} \{x \in X : |f_k(x) - f(x)| \geq \varepsilon\}\Big) \to 0, \quad n \to \infty.$$

Hint. Consider the set $\bigcup_{k=1}^{\infty} \bigcap_{n=1}^{\infty} \bigcup_{j=n}^{\infty} \{x \in X : |f_j(x) - f(x)| \geq \frac{1}{k}\}$.

b) Show that the convergence of the series $\sum_{n=1}^{\infty} \lambda(\{x \in X : |f_n(x) - f(x)| \geq \varepsilon\})$ for all $\varepsilon > 0$ implies that $f_n \to f$ λ-a.e.

▶ a) Let $A = \{x \in X : f_n(x) \not\to f(x)\}$ and

$$B := \bigcup_{k=1}^{\infty} \bigcap_{n=1}^{\infty} \bigcup_{j=n}^{\infty} \{x \in X : |f_j(x) - f(x)| \geq \tfrac{1}{k}\}.$$

We first show that $A = B$.

We recall the definition of convergence:

$a_n \to a$ iff $\forall \varepsilon > 0 \ \exists N \in \mathbb{N} \ \forall n \geq N \quad |a_n - a| < \varepsilon$.

This is equivalent to:

$\forall k > 0 \ \exists N \geq 1 \ \forall n \geq N \quad |a_n - a| < \frac{1}{k}$.

So, hence, $a_n \not\to a$ iff

$\exists k > 0 \ \forall N \geq 1 \ \exists n \geq N \quad |a_n - a| \geq \frac{1}{k}$.

Take $y \in A \Leftrightarrow f_n(y) \not\to f(y) \Leftrightarrow$

$\Leftrightarrow \exists k > 0 \ \forall N \geq 1 \ \exists n \geq N: \ |f_n(y) - f(y)| \geq \frac{1}{k} \Leftrightarrow$

$\Leftrightarrow \exists k > 0 \ \forall N \geq 1 \ x \in \bigcup_{n=N}^{\infty} \{x : |f_n(x) - f(x)| \geq \frac{1}{k}\} \Leftrightarrow$

$\Leftrightarrow \exists k > 0 \ x \in \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} \{x : |f_n(x) - f(x)| \geq \frac{1}{k}\} \Leftrightarrow$

$\Leftrightarrow x \in \bigcup_{k=1}^{\infty} \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} \{x : |f_n(x) - f(x)| \geq \frac{1}{k}\} \Leftrightarrow$

$\Leftrightarrow x \in B$. Hence $A = B$.

Show that $\forall \varepsilon > 0 \ \lambda\big(\bigcup_{k=n}^{\infty} \{x \in X : |f_k(x) - f(x)| \geq \varepsilon\}\big) \to 0, \ n \to \infty$.

Take an arbitrary $\varepsilon > 0$. Since $f_n \to f$ λ-a.e.,

$$0 = \lambda(\{f_n(x) \not\to f(x)\}) = \lambda(A) = \lambda(B) = \qquad (*)$$

$$= \lambda\Big(\bigcup_{k=1}^{\infty}\bigcap_{n=1}^{\infty}\bigcup_{j=n}^{\infty}\{x \in X : |f_j(x) - f(x)| \geq \tfrac{1}{k}\}\Big).$$

Remark that $\exists\, k \geq 1$ s.t. $\frac{1}{k} \leq \varepsilon$.

This implies that $\{x : |f_j(x) - f(x)| \geq \varepsilon\} \subseteq$

$$\subseteq \{x : |f_j(x) - f(x)| \geq \tfrac{1}{k}\}.$$

So,

$$\bigcap_{n=1}^{\infty}\bigcup_{j=n}^{\infty}\{x : |f_j(x) - f(x)| \geq \varepsilon\} \subseteq \bigcup_{k=1}^{\infty}\bigcap_{n=1}^{\infty}\bigcup_{j=n}^{\infty}\{x : |f_j(x) - f(x)| \geq \tfrac{1}{k}\}$$

This inclusion and $(*)$ imply that

$$\lambda\Big(\bigcap_{n=1}^{\infty}\underbrace{\bigcup_{j=n}^{\infty}\{x : |f_j(x) - f(x)| \geq \varepsilon\}}_{\text{decreases and } \lambda(\bigcup_{j=1}^{\infty}\{\dots\}) \leq \lambda(X) < +\infty}\Big) = 0$$

By the continuity of the measure

$$0 = \lambda\Big(\bigcap_{n=1}^{\infty}\bigcup_{j=n}^{\infty}\{x : |f_j(x) - f(x)| \geq \varepsilon\}\Big) =$$

$$= \lim_{n\to\infty}\lambda\Big(\bigcup_{j=n}^{\infty}\{x : |f_j(x) - f(x)| \geq \varepsilon\}\Big).$$

Next, let $\forall \varepsilon > 0$ $\lambda\Big(\bigcup_{k=n}^{\infty}\{x \in X : |f_k(x) - f(x)| \geq \varepsilon\}\Big) \to 0,\ n \to \infty.$

Then, by the continuity of the measure,

$$0 = \lim_{n\to\infty}\lambda\Big(\bigcup_{k=n}^{\infty}\{x : |f_k(x) - f(x)| \geq \varepsilon\}\Big) = \qquad (**)$$

$$= \lambda\Big(\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty}\{x : |f_k(x) - f(x)| \geq \varepsilon\}\Big).$$

Hence, using the σ-semiadditivity of measure:

$$\lambda\Big(\bigcup_{k=1}^{\infty}\bigcap_{n=1}^{\infty}\bigcup_{j=n}^{\infty}\{x : |f_j(x) - f(x)| \geq \varepsilon\}\Big) \leq$$

$$\leq \sum_{k=1}^{\infty}\underbrace{\lambda\Big(\bigcap_{n=1}^{\infty}\bigcup_{j=n}^{\infty}\{x : |f_j(x) - f(x)| \geq \varepsilon\}\Big)}_{=0 \text{ by } (**)} = 0 \qquad \blacktriangleleft$$

b) Let for every $\varepsilon > 0$

$$\sum_{n=1}^{\infty} \lambda(\{x : |f_n(x) - f(x)| \geq \varepsilon\}) \text{ converges.}$$

Take $\varepsilon > 0$ and estimate

$$\lambda\left(\bigcup_{k=n}^{\infty} \{x : |f_n(x) - f(x)| \geq \varepsilon\}\right) \leq$$

$$\leq \sum_{k=n}^{\infty} \lambda(\{x : |f_n(x) - f(x)| \geq \varepsilon\}) \xrightarrow[n\to\infty]{} 0,$$

as the tail of convergent series.

By a) $f_n \to f$ λ-a.e. ◀