1 (i) Let $A \in \mathcal{F}$ and for every $n \geq 1$, functions $f_n \in L(A,\lambda)$
[3p.] and are non-negative. Let also the series $\sum_{n=1}^{\infty} f_n$ converges λ-a.e. on A. Show that

$$\sum_{n=1}^{\infty} \int_A f_n \, d\lambda = \int_A \left(\sum_{n=1}^{\infty} f_n\right) d\lambda.$$

▶ Consider $S_n(x) = \sum_{k=1}^{n} f_k(x)$. Since f_n are non-negative, then $0 \leq S_n(x) \leq S_{n+1}(x)$, $\forall n \geq 1$, $x \in A$. By the monotone convergence theorem

$$\lim_{n\to\infty} \int_A S_n \, d\lambda = \int_A \lim_{n\to\infty} S_n \, d\lambda. \qquad (*)$$

Since $\sum_{n=1}^{\infty} f_n$ converges λ-a.e on A, then the sequence S_n, $n \geq 1$, converges λ-a.e and $\lim_{n\to\infty} S_n$ exists.

Hence,

$$\int_A \sum_{n=1}^{\infty} f_n \, d\lambda = \int_A \left(\lim_{n\to\infty} S_n\right) d\lambda \overset{\text{by } (*)}{=} \lim_{n\to\infty} \int_A S_n \, d\lambda =$$

$$= \lim_{n\to\infty} \int_A \sum_{k=1}^{n} f_k \, d\lambda = \lim_{n\to\infty} \sum_{k=1}^{n} \int_A f_k \, d\lambda = \sum_{n=1}^{\infty} \int_A f_n \, d\lambda.$$

by the property of integral

◀

(1 ii) Let $X=[0;1]$, $\mathcal{F}=\mathcal{B}([0;1])$, λ be a Lebesgue measure on $[0;1]$. Consider the integrable functions

$$f_1 = 2\,\mathbb{I}_{[0;\frac{1}{2}]}, \quad f_n = \mathbb{I}_{[0;\frac{1}{n+1}]} - n\,\mathbb{I}_{(\frac{1}{n+1};\frac{1}{n}]}, \quad n \geq 2.$$

Show that the series $\sum_{n=1}^{\infty} f_n$ converges a.e. on $[0;1]$, but

$$\sum_{n=1}^{\infty} \int_0^1 f_n\,dx \neq \int_0^1 \left(\sum_{n=1}^{\infty} f_n\right) dx.$$

▶

$$f_1(x) = 2\,\mathbb{I}_{[0;\frac{1}{2}]}(x) = 2 \cdot \begin{cases} 1, & x \in [0;\frac{1}{2}] \\ 0, & x \in (\frac{1}{2};1] \end{cases} = \begin{cases} 2, & x \in [0;\frac{1}{2}] \\ 0, & x \in (\frac{1}{2};1]. \end{cases}$$

$$f_n(x) = \mathbb{I}_{[0;\frac{1}{n+1}]} - n\cdot\mathbb{I}_{(\frac{1}{n+1};\frac{1}{n}]} = \begin{cases} 1, & x \in [0;\frac{1}{n+1}] \\ 0, & x \in (\frac{1}{n+1};1] \end{cases} - \begin{cases} n, & x \in (\frac{1}{n+1};\frac{1}{n}] \\ 0, & x \in [0;\frac{1}{n+1}] \cup (\frac{1}{n};1] \end{cases} =$$

$$= \begin{cases} 1, & x \in [0;\frac{1}{n+1}] \\ -n, & x \in (\frac{1}{n+1};\frac{1}{n}] \\ 0, & x \in (\frac{1}{n};1], \end{cases} \quad n \geq 2.$$

Show that the series $\sum_{n=1}^{\infty} f_n$ converges a.e. on $[0;1]$.

Compute $S_n(x) = \sum_{k=1}^{n} f_k(x)$ and show that S_n converge a.e. on $[0;1]$.

$$f_1(x) = \begin{cases} 2, & x \in [0;\frac{1}{2}] \\ 0, & x \in (\frac{1}{2};1] \end{cases} \qquad f_2(x) = \begin{cases} 1, & x \in [0;\frac{1}{3}] \\ -2, & x \in (\frac{1}{3};\frac{1}{2}] \\ 0, & x \in (\frac{1}{2};1] \end{cases}$$

Then

$$f_1(x) + f_2(x) = \begin{cases} 3, & x \in [0;\frac{1}{3}] \\ 0, & x \in (\frac{1}{3};\frac{1}{2}] \\ 0, & x \in (\frac{1}{2};1] \end{cases} = \begin{cases} 3, & x \in [0;\frac{1}{3}] \\ 0, & x \in (\frac{1}{3};1]. \end{cases}$$

$$f_3(x) = \begin{cases} 1, & x \in [0;\frac{1}{4}] \\ -3, & x \in (\frac{1}{4};\frac{1}{3}] \\ 0, & x \in (\frac{1}{3};1] \end{cases}$$

Then

$$f_1(x) + f_2(x) + f_3(x) = \begin{cases} 4, & x \in [0;\frac{1}{4}] \\ 0, & x \in (\frac{1}{4};1] \end{cases}$$

and

$$S_n(x) = f_1(x) + f_2(x) + \ldots + f_n(x) =$$

$$= \begin{cases} n+1, & x \in [0; \frac{1}{n+1}] \\ 0, & x \in (\frac{1}{n+1}; 1] \end{cases} \xrightarrow[n \to \infty]{} \begin{cases} \infty, & x = 0 \\ 0, & x \in (0; 1]. \end{cases}$$

Hence,

$$\lim_{n\to\infty} S_n(x) = 0 \quad \lambda\text{-a.e on } [0;1], \quad \varphi = \{0\}, \ \lambda(\varphi) = 0.$$

$\sum_{n=1}^{\infty} f_n(x) < +\infty$ and then

$$\int_0^1 \Big(\underbrace{\sum_{n=1}^{\infty} f_n(x)}_{=0}\Big) dx = \int_0^1 0 \, dx = 0. \qquad (*)$$

Show that

$$\sum_{n=1}^{\infty} \int_0^1 f_n(x) dx \neq \int_0^1 \Big(\sum_{n=1}^{\infty} f_n(x)\Big) dx.$$

For this compute $\int_0^1 f_n(x) dx$.

If $n = 1$

$$\int_0^1 f_1(x) dx = \int_{[0;\frac{1}{2}]} 2 \, dx + \int_{(\frac{1}{2}; 1]} 0 \, dx = 2 \cdot \lambda([0; \tfrac{1}{2}]) = 1.$$

If $n \geq 2$

$$\int_0^1 f_n(x) dx = \int_{[0;\frac{1}{n+1}]} \underbrace{f_n(x)}_{=1} dx + \int_{(\frac{1}{n+1}; \frac{1}{n}]} \underbrace{f_n(x)}_{=-n} dx + \int_{(\frac{1}{n}; 1]} \underbrace{f_n(x)}_{=0} dx =$$

$$= 1 \cdot \frac{1}{n+1} - n \cdot \Big(\frac{1}{n} - \frac{1}{n+1}\Big) + 0 \cdot \Big(1 - \frac{1}{n}\Big) = \frac{1}{n+1} - \frac{n(n+1-n)}{n(n+1)} =$$

$$= \frac{1}{n+1} - \frac{1}{n+1} = 0.$$

Then

$$\sum_{n=1}^{\infty} \int_0^1 f_n(x) dx = \underbrace{\int_0^1 f_1(x) dx}_{=1} + \underbrace{\sum_{n=2}^{\infty} \int_0^1 f_n(x) dx}_{=0} = 1. \qquad (**)$$

Hence, we have by $(*)$ and $(**)$

$$0 = \int_0^1 \Big(\sum_{n=1}^{\infty} f_n(x)\Big) dx \neq \sum_{n=1}^{\infty} \int_0^1 f_n(x) dx = 1$$

◀

② Let p_k, $k \geq 1$, be a non-negative numbers satisfying the condition

$$\sup_{0<s<1} \sum_{k=1}^{\infty} \frac{\sin^2(sk)}{s^2} p_k < +\infty.$$

Prove that $\sum_{k=1}^{\infty} k^2 p_k < +\infty$.

▶ Set $X = \mathbb{N}$, $\lambda(\{k\}) = p_k$, $k \in \mathbb{N}$.

Let $f_s(k) = \frac{\sin^2(sk)}{s^2}$, $k \in \mathbb{N}$.

Note that $f_s \geq 0$ $\forall s \in (0;1)$.

Take $f(k) = k^2$, $k \in \mathbb{N}$. We show that $\lim_{s\to 0} f_s(k) = \lim_{s\to 0} \frac{\sin^2(sk)}{s^2}$

$= \lim_{s\to 0} \underbrace{\frac{\sin^2(sk)}{(sk)^2}}_{\xrightarrow[s\to 0]{} 1} \cdot k^2 = k^2.$

By the Fatou's lema,

$$\sum_{k=1}^{\infty} k^2 p_k = \int_{\mathbb{N}} f(k)\,\lambda(dk) = \int_{\mathbb{N}} \underbrace{\lim_{s\to 0} f_s(k)}_{= f(k)}\,\lambda(dk) =$$

$$= \int_{\mathbb{N}} \varliminf_{s\to 0} f_s(k)\,\lambda(dk) \overset{\text{Fatou's l.}}{\leq} \varliminf_{s\to 0} \int_{\mathbb{N}} f_s(k)\,\lambda(dk) =$$

$$= \varliminf_{s\to 0} \sum_{k=1}^{\infty} f_s(k) \cdot p_k \leq \sup_{0<s<1} \sum_{k=1}^{\infty} \frac{\sin^2(sk)}{s^2} p_k \underset{\text{given}}{<} +\infty.$$

Hence, $\sum_{k=1}^{\infty} k^2 p_k < +\infty$. ◀

(3) Let $f:[0;1]\to\mathbb{R}$ be non-negative and Borel measurable. Compute
[4p]
$$\lim_{n\to\infty}\int_0^1 x^{nf(x)}\,dx.$$

▶ Let $g_n(x)=x^{nf(x)}$, $x\in X := [0;1]$.

For computation let us use the dominated convergence theorem. For this check the following conditions:

1) $g_n\to g$ λ-a.e on X;
2) $\exists\, h\in L(X,\lambda)$: $|g_n(x)|\le h(x)$ $\forall n\ge 1$, $\forall x\in X$.

Then $g, g_n\in L(X,\lambda)$, $n\ge 1$ and
$$\lim_{n\to\infty}\int_0^1 g_n(x)\,dx=\int_0^1 g(x)\,dx.$$

1) Let $A=\{x\in[0;1]: f(x)>0\}$. Then
$$\lim_{n\to\infty} g_n(x)=\lim_{n\to\infty} x^{nf(x)}=\begin{cases}0, & x\in A,\\ 1, & x\notin A,\end{cases}\quad x\in[0;1).$$

Set $g(x)=\begin{cases}0, & x\in A,\\ 1, & x\notin A,\end{cases}$ $x\in[0;1)$.

Then $g_n\to g$ λ-a.e. $(\varphi=\{1\})$.

2) $\forall x\in X$, $\forall n\ge 1$ $|g_n(x)|=|x^{nf(x)}|\le \underset{=h(x)}{1}$ and
$$\int_0^1 h(x)\,dx=\int_0^1 1\,dx=1.$$

Hence, $g_n, g\in L(X,\lambda)$ and
$$\lim_{n\to\infty}\int_0^1 g_n(x)\,dx=\int_0^1\lim_{n\to\infty}g_n(x)\,dx=\int_A \underset{=0}{g(x)}\,dx+\int_{[0;1]\setminus A}\underset{=1}{g(x)}\,dx=$$
$$=\int_{[0;1]\setminus A}1\,dx=\lambda([0;1]\setminus A)=\lambda(\{x\in \underset{=[0;1]}{X}: f(x)=0\}).$$
◀

④ Compute the sum $\sum_{n=1}^{\infty} \int_1^{+\infty} \frac{dx}{(1+x^2)^n}$.
[3p]

▸ Set $f_n(x) = \frac{1}{(1+x^2)^n}$.
Show that for every $n \geq 1$ functions $f_n \in L(X, \lambda)$ and are non-negative. Then if $\sum_{n=1}^{\infty} f_n$ converges on $X = [1; +\infty)$, then

$$\sum_{n=1}^{\infty} \int_1^{+\infty} \frac{dx}{(1+x^2)^n} = \int_1^{+\infty} \sum_{n=1}^{\infty} \frac{1}{(1+x^2)^n}\, dx. \qquad (*)$$

1) $f_n(x) = \frac{1}{(1+x^2)^n} > 0 \quad \forall n \geq 1.$

2) $\int_1^{+\infty} \frac{dx}{(1+x^2)^n} < +\infty$, because

$\forall x \in [1; +\infty) \quad \left|\frac{1}{(1+x^2)^n}\right| \leq \frac{1}{x^{2n}}$ and

$$\int_1^{+\infty} \frac{1}{x^{2n}}\, dx = \frac{1}{(-2n+1)x^{2n-1}}\Big|_1^{+\infty} = \frac{1}{2n-1} < +\infty.$$

3) Show that $\sum_{n=1}^{\infty} \frac{1}{(1+x^2)^n}$ converges λ-a.e. We know that

$\sum_{n=0}^{\infty} q^n = \frac{1}{1-q}$, $|q| < 1$. Then

$$\sum_{n=1}^{\infty} \frac{1}{(1+x^2)^n} = \sum_{n=0}^{\infty} \frac{1}{(1+x^2)^n} - 1 = \frac{1}{1-\frac{1}{1+x^2}} - 1 =$$

$$= \frac{1+x^2}{1+x^2-1} - 1 = \frac{1+x^2}{x^2} - 1 = \frac{1}{x^2} + 1 - 1 = \frac{1}{x^2} < +\infty \quad \forall x \in [1; +\infty)$$

Hence,

$$\sum_{n=1}^{\infty} \int_1^{+\infty} \frac{dx}{(1+x^2)^n} \overset{\text{by } (*)}{=} \int_1^{+\infty} \underbrace{\sum_{n=1}^{\infty} \frac{1}{(1+x^2)^n}}_{=\frac{1}{x^2}}\, dx = \int_1^{+\infty} \frac{1}{x^2}\, dx = \left(-\frac{1}{x}\right)\Big|_1^{+\infty} = 1.$$

◀

(5) Let F be a continuously differentiable non-decreasing function on $\mathbb{R}$ with $F'=f$. Show that

$$\int_A g(x)\,dF(x) = \int_A g(x) f'(x)\,dx$$

for every non-negative function g on $\mathbb{R}$ and a Borel set A.

▶ 1) We first show that $\forall A \in \mathcal{B}(\mathbb{R})$

$$\int_A 1\,dF(x) = \int_A f(x)\,dx.$$

Set $\mu_1(A) := \int_A 1\,dF(x) = \lambda_F(A)$, $\mu_2(A) := \int_A f(x)\,dx$.

Remark that μ_1 and μ_2 are measures on $\mathcal{B}(\mathbb{R})$. Moreover, they coinside on the semiring

$$H = \{(a;b] : a<b\} \cup \{\emptyset\}.$$

Indeed $\mu_1((a,b]) = \lambda_F(A) = F(b) - F(a) =$

$$= \int_a^b f(x)\,dx = \mu_2((a;b])$$

by the fundamental theorem of calculus. By the uniqueness of the extension of a measure from a semiring to the generated σ-algebra, we have that

$$\mu_1(A) = \mu_2(A) \quad \forall A \in \mathcal{B}(\mathbb{R}).$$

2) Let $p : \mathbb{R} \to [0;+\infty)$ be a simple function, given by $p(x) = \sum_{k=1}^n a_k \mathbb{I}_{A_k}$, $A_k \cap A_j = \emptyset$, $\bigcup_{k=1}^n A_k = \mathbb{R}$.

Then

$$\int_A p(x)\,dF = \sum_{n=1}^n a_k \int_A \mathbb{I}_{A_k}(x)\,dF(x) =$$

$$= \sum_{k=1}^n a_k \int_{A\cap A_k} dF(x) = \sum_{k=1}^n a_k \int_{A\cap A_k} f(x)\,dx = \sum_{k=1}^n a_k \int_A f(x)\mathbb{I}_{A_k}(x)\,dx =$$

$$= \int_A f(x)\Big(\sum_{k=1}^n a_k \mathbb{I}_{A_k}(x)\Big)dx = \int_A p(x) f(x)\,dx.$$

3) Let $g \geq 0$ be any measurable function.

Let g_n be a sequence of simple functions s.t.

$$0 \leq g_n(x) \leq g_{n+1}(x), \quad x \in \mathbb{R},$$

and

$$g_n(x) \to g(x) \quad \forall x \in \mathbb{R}.$$

(See Theorem 7.4).

Then, by the monotone convergence theorem

$$\int_A g \, dF = \lim_{n\to\infty} \int_A g_n \, dF = \lim_{n\to\infty} \int_A g_n f \, dx = \int_A g f \, dx.$$

▲

⑥ Let $\ell_n^\infty := \mathbb{R}^n$ and $d(x,y) = \max\limits_{k=1,\dots,n} |\xi_k - \eta_k|$,
[2p] $x = (\xi_k)_{k=1}^n$, $y = (\eta_k)_{k=1}^n \in \ell_n^\infty$.
Show that (ℓ_n^∞, d) is a metric space.

▶ Prove that d is a metric. For this we need to check the following conditions:

(M1) $d(x,y) \geq 0$;
(M2) $d(x,y) = 0 \Leftrightarrow x = y$;
(M3) $d(x,y) = d(y,x)$;
(M4) $d(x,y) \leq d(x,z) + d(z,y)$, $x,y,z \in \mathbb{R}^n$.

(M1): $d(x,y) \geq 0$, because $|\xi_k - \eta_k| \geq 0 \;\; \forall k = 1,\dots,n \Rightarrow$
$\Rightarrow \max\limits_{k=1,\dots,n} |\xi_k - \eta_k| \geq 0 \;\; \forall k = 1,\dots,n$.

(M2): $d(x,y) = 0 \Leftrightarrow \max\limits_{k=1,\dots,n} |\xi_k - \eta_k| = 0 \Leftrightarrow |\xi_k - \eta_k| = 0 \Leftrightarrow$
$\Leftrightarrow \xi_k = \eta_k \;\; \forall k = 1,\dots,n \Leftrightarrow x = y$.

(M3): $d(x,y) = \max\limits_{k=1,\dots,n} |\xi_k - \eta_k| = \max\limits_{k=1,\dots,n} |\eta_k - \xi_k| = d(y,x)$.

(M4): let $z = (\zeta_k)_{k=1}^n$.

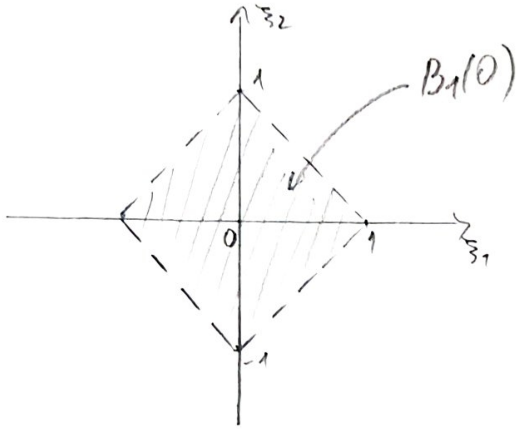
$$d(x,y) = \max_{k=1,\dots,n} |\xi_k - \eta_k| = \max_{k=1,\dots,n} |\xi_k - \zeta_k + \zeta_k - \eta_k| \leq$$
$$\leq \max_{k=1,\dots,n} \left(|\xi_k - \zeta_k| + |\zeta_k - \eta_k|\right) \leq$$
$$\leq \max_{k=1,\dots,n} |\xi_k - \zeta_k| + \max_{k=1,\dots,n} |\zeta_k - \eta_k| =$$
$$= d(x,z) + d(z,y).$$

Hence, d is a metric on $\ell_n^\infty \Rightarrow (\ell_n^\infty, d)$ is a metric space. ◀

(7) Draw the balls $B_1(0)$ in the following metric spaces ℓ_2^1, ℓ_2^2 and ℓ_2^∞.

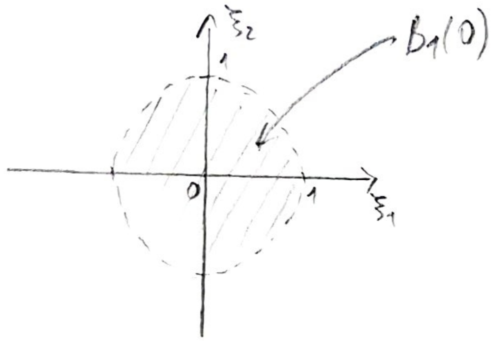
- ℓ_2^1 : $d(x,y) = |\xi_1 - \eta_1| + |\xi_2 - \eta_2|$, $x = (\xi_1, \xi_2)$, $y = (\eta_1, \eta_2)$

$$B_1(0) = \{x \in \mathbb{R}^2 : d(x,0) < 1\} = \{x \in \mathbb{R}^2 : |\xi_1 - 0| + |\xi_2 - 0| < 1\} =$$
$$= \{x \in \mathbb{R}^2 : |\xi_1| + |\xi_2| < 1\}.$$



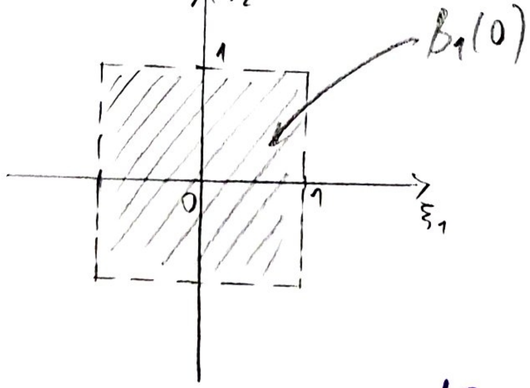
ℓ_2^2 : $d(x,y) = \left(|\xi_1 - \eta_1|^2 + |\xi_2 - \eta_2|^2\right)^{\frac{1}{2}}$.

$$B_1(0) = \{x \in \mathbb{R}^2 : d(x,0) < 1\} = \{x \in \mathbb{R}^2 : (\xi_1^2 + \xi_2^2)^{1/2} < 1\} =$$
$$= \{x \in \mathbb{R}^2 : \xi_1^2 + \xi_2^2 < 1\}.$$



ℓ_2^∞ : $d(x,y) = \max(|\xi_1 - \eta_1|, |\xi_2 - \eta_2|)$.

$$B_1(0) = \{x \in \mathbb{R}^2 : d(x,0) < 1\} = \{x \in \mathbb{R}^2 : \max(|\xi_1|, |\xi_2|) < 1\}$$



b) $f(x) = \cos x, \quad g(x) = \sin x, \quad x \in [0; 2\pi]$

$L_2[0;2\pi]$: $\quad d(f,g) = d(\cos x, \sin x) =$

$$= \left(\int_0^{2\pi} |\cos x - \sin x|^2 \, dx \right)^{1/2} =$$

$$= \left(\int_0^{2\pi} (\cos x - \sin x)^2 \, dx \right)^{\frac{1}{2}} = \left(\int_0^{2\pi} (\cos^2 x - 2\sin x \cdot \cos x + \sin^2 x) dx \right)^{\frac{1}{2}} =$$

$$= \left(\int_0^{2\pi} (1 - 2\sin x \cdot \cos x) \, dx \right)^{1/2} = \left(\int_0^{2\pi} (1 - \sin 2x) \, dx \right)^{\frac{1}{2}} =$$

$$= \left(\left(x + \frac{\cos 2x}{2} \right) \Big|_0^{2\pi} \right)^{1/2} = \left(\left(2\pi + \frac{1}{2}\right) - \left(0 + \frac{1}{2}\right) \right)^{\frac{1}{2}} = \sqrt{2\pi}.$$

Hence,

$$d(\cos x, \sin x) = \sqrt{2\pi}.$$

$\blacktriangle$

⑧ Compute the distances between the functions $\cos x$ and $\sin x$, $x \in [0; 2\pi]$, in
a) $C[0;2\pi]$ and b) $L_2[0;2\pi]$.

▸ a) $f(x) = \cos x$, $g(x) = \sin x$, $x \in [0; 2\pi]$.

$$d(f,g) = \max_{x\in[0;2\pi]} |f(x) - g(x)| = \max_{x\in[0;2\pi]} |\cos x - \sin x|.$$

Let $h(x) = \cos x - \sin x$. Then $d(f,g) = \max\limits_{x\in[0;2\pi]} |h(x)| =$

$$= \max(|h_{max}|, |h_{min}|).$$

We need to find h_{max} and h_{min}.

$h(x) = \cos x - \sin x$,

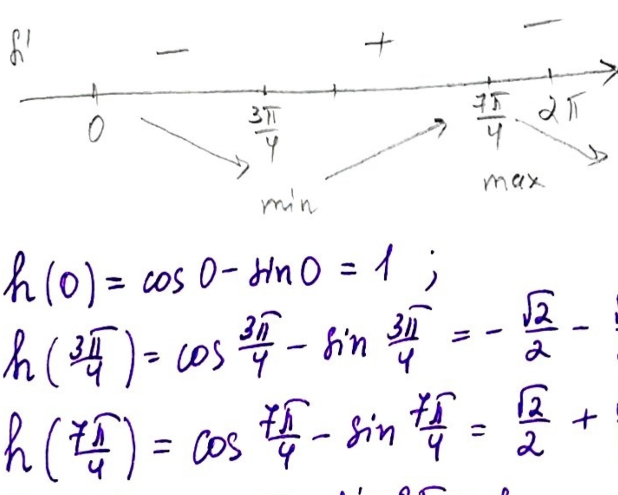
$h'(x) = -\sin x - \cos x$,

$h'(x) = 0 \Leftrightarrow -\sin x - \cos x = 0 \quad |: \cos x \neq 0$

$\operatorname{tg} x = -1$

$x = -\frac{\pi}{4} + \pi n, \quad n \in \mathbb{Z}$

Since $x \in [0; 2\pi]$, then take $x_1 = \frac{3\pi}{4}$ and $x_2 = \frac{7\pi}{4}$



$$\left.\begin{aligned} &h(0) = \cos 0 - \sin 0 = 1; \\ &h\left(\tfrac{3\pi}{4}\right) = \cos\tfrac{3\pi}{4} - \sin\tfrac{3\pi}{4} = -\tfrac{\sqrt{2}}{2} - \tfrac{\sqrt{2}}{2} = -\sqrt{2}; \\ &h\left(\tfrac{7\pi}{4}\right) = \cos\tfrac{7\pi}{4} - \sin\tfrac{7\pi}{4} = \tfrac{\sqrt{2}}{2} + \tfrac{\sqrt{2}}{2} = \sqrt{2}; \\ &h(2\pi) = \cos 2\pi - \sin 2\pi = 1. \end{aligned}\right\} \Rightarrow \begin{aligned} h_{max} &= \sqrt{2} \\ h_{min} &= -\sqrt{2} \end{aligned}$$

Hence, $d(f,g) = d(\cos x, \sin x) = \max(|h_{max}|, |h_{min}|) =$

$$= \max(|\sqrt{2}|, |-\sqrt{2}|) = \sqrt{2}.$$

◀