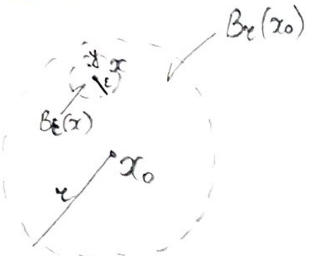
(1) Justify the terms „open ball" and „closed ball" by proving that

a) any open ball is an open set;

b) any closed ball is a closed set.

▶ Let (X, d) be a metric spase.

a) The set $B_r(x_0) = \{x \in X : d(x, x_0) < r\}$ is called an open ball.



We need to show that any open ball $B_r(x_0)$ is an open set.
$B_r(x_0)$ is an open set if for $\forall x \in B_r(x_0)$ $\exists \varepsilon > 0$ s.t. $B_\varepsilon(x) \subset B_r(x_0)$.

Take $\forall x \in B_r(x_0)$ and show that $B_\varepsilon(x) \subset B_r(x_0)$.
Set $\varepsilon = \frac{r - d(x, x_0)}{2}$. Then for $\forall y \in B_\varepsilon(x)$ we have

$$d(x, y) < \varepsilon = \frac{r - d(x, x_0)}{2}$$

and

$$d(y, x_0) \underbrace{\leq}_{\text{triangle ineq.}} \underbrace{d(y, x)}_{< \frac{r - d(x, x_0)}{2}} + \underbrace{d(x, x_0)}_{< r} < \frac{r - d(x, x_0)}{2} + d(x, x_0) = \frac{r + \overbrace{d(x, x_0)}^{< r}}{2} <$$

$$< \frac{r + r}{2} = r \Rightarrow y \in B_r(x_0). \Rightarrow B_\varepsilon(x) \subset B_r(x_0).$$

We showed that for $\forall y \in B_\varepsilon(x)$ $\exists \varepsilon > 0$ s.t. $B_\varepsilon(x) \subset B_r(x_0)$.
Hence, $B_r(x_0)$ is an open set. ◀

b) The set $\bar{B}_r(x_0) = \{x \in X : d(x, x_0) \le r\}$ is called a closed ball.

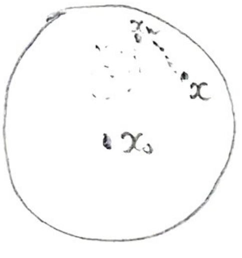
We need to show that any closed ball is a closed set.

$\bar{B}_r(x_0)$ is closed if and only if for any sequence $x_n \in \bar{B}_r(x_0)$, $n \ge 1$, that converges to $x \in X$ one has that $x \in \bar{B}_r(x_0)$.

Take an arbitrary sequence $x_n \in \bar{B}_r(x_0)$, $n \ge 1$, s.t. $x_n \to x$.

We need to show that $x \in \bar{B}_r(x_0)$, that is

$$d(x, x_0) \le r. ?$$



Since $x_n \in \bar{B}_r(x_0)$, then

$$d(x_n, x_0) \le r.$$

Let us use the inequality

$$|d(x, x_0) - d(x, x_n)| \le d(x_n, x_0)$$

$$-d(x_n, x_0) \le d(x, x_0) - d(x, x_n) \le d(x_n, x_0)$$

$$0 \le d(x, x_0) \le \underbrace{d(x_n, x_0)}_{\le r} + \underbrace{d(x, x_n)}_{\to 0}$$

Since $x_n \to x$, then $d(x, x_n) \to 0$, $n \to \infty$.

$\Rightarrow d(x, x_0) \le r$.

Hence, $x \in \bar{B}_r(x_0)$ and $\bar{B}_r(x_0)$ is a closed set.

② Check if the following sets are open in $C[0;2]$.

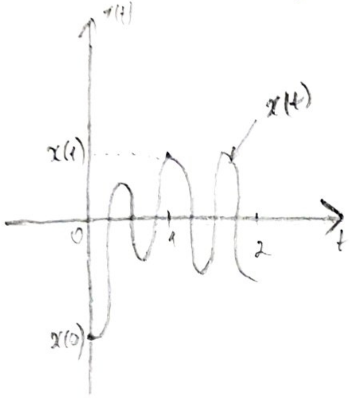
a) $A = \{x \in C[0;2]: \ x(0) < 0, \ x(1) > 0\}$;

б) $B = \{x \in C[0;2]: \ \int_0^2 |x(t)|\,dt < 1\}$.

▸ a) $A = \{x \in C[0;2]: \ x(0) < 0, \ x(1) > 0\}$.

We need to prove that A is an open set in $C[0;2]$.

A is an open set if for $\forall x \in A \ \exists \varepsilon > 0$ s.t. $B_\varepsilon(x) \subset A$.



Take $\varepsilon = \min\{\underbrace{-\underbrace{x(0)}_{<0}}_{>0}, \underbrace{x(1)}_{>0}\} > 0$ and

show that $B_\varepsilon(x) \subset A$, that is

for $\forall y \in B_\varepsilon(x)$ we need to check that

$y \in A$ $(y(0) < 0, \ y(1) > 0)$.

Since $y \in B_\varepsilon(x)$, then $d(x,y) < \varepsilon$.

$$d(x,y) = \max_{t \in [0,2]} |x(t) - y(t)| < \varepsilon \ \Rightarrow$$

$$\Rightarrow \ |x(t) - y(t)| < \varepsilon \qquad \forall t \in [0;2]$$

$$-\varepsilon + x(t) < y(t) < \varepsilon + x(t). \qquad (*)$$

$$\begin{cases} y(0) \overset{(*)}{<} \varepsilon + x(0) \le -x(0) + x(0) = 0 \ \left(\varepsilon = \min\{-x(0), x(1)\} \Rightarrow \varepsilon \le -x(0) \text{ and } \begin{matrix} \varepsilon \le x(1) \\ -\varepsilon \ge -x(1) \end{matrix} \right) \\ y(1) > -\varepsilon + x(1) \ge -x(1) + x(1) = 0 \end{cases}$$

$$\begin{matrix} y(0) < 0 \\ y(1) > 0 \end{matrix} \ \Rightarrow \ y \in A.$$

Hence, A is an open set in $C[0;2]$. ◀

b) $B = \{x \in C[0;2] : \int_0^2 |x(t)|dt < 1\}$.

We need to prove that B is an open set in $C[0;2]$.

For $\forall x \in B$ we need to find $\varepsilon > 0$ s.t. $B_\varepsilon(x) \subset B$.

Take $y \in B_\varepsilon(x)$. Then $d(x,y) = \max\limits_{t\in[0;2]} |x(t) - y(t)| < \varepsilon$.

Let us find $\varepsilon > 0$ s.t. $y \in B(\int_0^2 |y(t)|dt < 1)$

$$\int_0^2 |y(t)|dt = \int_0^2 |y(t) - x(t) + x(t)|dt \leq$$

$$\leq \int_0^2 \underbrace{|y(t) - x(t)|}_{<\varepsilon} dt + \int_0^2 |x(t)|dt <$$

$$< 2\varepsilon + \int_0^2 |x(t)|dt. \Rightarrow \int_0^2 |y(t)|dt < 1 \quad \text{if}$$

$$2\varepsilon + \int_0^2 |x(t)|dt \leq 1$$

$$\Rightarrow \varepsilon = \frac{1}{2}\left(1 - \int_0^2 |x(t)|dt\right).$$

Hence, $\forall x \in B \ \exists \varepsilon = \frac{1}{2}\left(1 - \int_0^2 |x(t)|dt\right)$ s.t.

$$B_\varepsilon(x) \subset B$$

$\Rightarrow B$ is an open set in $C[0;2]$ ∎

③ Prove that the space ℓ_n^p is separable for every $p \geq 1$.

▶ $X = \ell_n^p = \mathbb{R}^n$.

Let $x = (\xi_1, \dots, \xi_n)$, $y = (\eta_1, \dots, \eta_n) \in X$,

$d(x,y) = \left(\sum_{k=1}^{n} |\xi_k - \eta_k|^p\right)^{1/p}$.

We need to show that ℓ_n^p is separable.

Recall that X is separable if there exists a countable set $M \subseteq X$ s.t. every ball $B_\varepsilon(x)$, $\varepsilon > 0$, $x \in X$, contains points from M.

Let $M = \{x \in \ell_n^p : x = (\xi_1, \dots, \xi_n),\ \xi_k \in \mathbb{Q},\ k = \overline{1,n}\}$.

M is countable.

Take $\forall\, y = (\eta_1, \dots, \eta_n) \in \ell_n^p$ and $\varepsilon > 0$ and find $x \in M$ s.t. $x \in B_\varepsilon(y)$.

$$x \in B_\varepsilon(y) \iff d(x,y) < \varepsilon.$$

Choose $\xi_1, \dots, \xi_n$ s.t.

$$|\xi_k - \eta_k| < \frac{\varepsilon}{\sqrt[p]{n}}, \quad k = 1, \dots, n.$$

Then

$$d(x,y) = \Big(\sum_{k=1}^{n} \underbrace{|\xi_k - \eta_k|^p}_{< \frac{\varepsilon}{\sqrt[p]{n}}}\Big)^{1/p} < \left(\sum_{k=1}^{n} \frac{\varepsilon^p}{n}\right)^{1/p} = \Big(\varepsilon^p \underbrace{\sum_{k=1}^{n} \frac{1}{n}}_{=1}\Big)^{1/p} = \varepsilon$$

$\Rightarrow d(x,y) < \varepsilon$.

Hence, ℓ_n^p is separable. ◀

④ Using the definition, show that the map $T: \ell^\infty \to \ell_2^p$ defined by the equality

$$Tx = (\xi_1, \xi_3), \quad x = (\xi_k)_{k=1}^\infty \in \ell^\infty$$

is continuous for every $p \geq 1$.

▶ A map $T: X \to Y$ is continuous on X if T is continuous at every point $x \in X$, that is,

$$\forall \varepsilon > 0 \ \exists \delta > 0 : \forall y \in X \ \ d_X(x,y) < \delta \Rightarrow d_Y(Tx, Ty) < \varepsilon.$$

$T: X \to Y$, $X = \ell^\infty$, $Y = \ell_2^p$.

Let $x = (\xi_k)_{k=1}^\infty \in \ell^\infty$ and take $\varepsilon > 0$.

We need to find $\underline{\delta > 0}$ s.t. $d_{\ell^\infty}(x,y) < \delta \Rightarrow d_{\ell_2^p}(Tx, Ty) < \varepsilon$, $\forall y \in \ell^\infty$.

Recall that $d_{\ell^\infty}(x,y) = \sup_k |\xi_k - \eta_k| < \delta$.

Then

$$d_{\ell_2^p}(Tx, Ty) = d_{\ell_2^p}\big((\xi_1, \xi_3), (\eta_1, \eta_3)\big) = \big(\underbrace{|\xi_1 - \eta_1|}_{<\delta}^p + \underbrace{|\xi_3 - \eta_3|}_{<\delta}^p\big)^{1/p} <$$

$$< (\delta^p + \delta^p)^{1/p} = (2\delta^p)^{1/p} = 2^{\frac{1}{p}} \delta = \varepsilon \Rightarrow$$

$$\Rightarrow \delta = \frac{\varepsilon}{\sqrt[p]{2}}.$$

Hence, for $x \in \ell^\infty$ and $\forall \varepsilon > 0$ exists $\delta = \frac{\varepsilon}{\sqrt[p]{2}} > 0$ s.t.

$\forall y \in \ell^\infty \quad d_{\ell^\infty}(x,y) < \delta \Rightarrow d_{\ell_2^p}(Tx, Ty) < \varepsilon \Rightarrow$

$\Rightarrow T$ is continuous at all $x \in \ell^\infty \Rightarrow$

$\Rightarrow T$ is continuous on ℓ^∞. ◀

⑤ If $\{x_n\}_{n\geq 1}$ is a Cauchy sequence in X and has a convergent subsequence, say, $x_{n_k} \to x$. Show that $\lim\limits_{n\to\infty} x_n = x$.

▶ $\{x_n\}_{n\geq 1}$ is a Cauchy sequence in X $\Leftrightarrow$
$\forall \varepsilon > 0 \quad \exists N \geq 1 \quad \forall n, m \geq N \quad d(x_n, x_m) < \frac{\varepsilon}{2}$.

A subsequence $\{x_{n_k}\}_{k\geq 1}$ converges to x $\Leftrightarrow$
$\forall \varepsilon > 0 \quad \exists K \geq 1 \quad \forall k \geq K \quad d(x_{n_k}, x) < \frac{\varepsilon}{2}$.

We need to show that $x_n \to x$, $n \to \infty$, that is, we need to prove that
$\forall \varepsilon > 0 \quad \exists \tilde{K} \geq 1 \quad \forall n \geq \tilde{K} \quad d(x_n, x) < \varepsilon$.

Take $\varepsilon > 0$ and $\tilde{K} = \max(N, K)$. Then for $\forall n, k \geq \tilde{K}$ we have

- $d(x_n, x_{n_k}) < \frac{\varepsilon}{2}$, since $n \geq \tilde{K} = \max(N, K) \geq N$,
  $n_k \geq k \geq \tilde{K} = \max(N, K) \geq N$.
  defin. of Cauchy seq.
- $d(x_{n_k}, x) < \frac{\varepsilon}{2}$, since $k \geq \tilde{K} = \max(N, K) \geq K$.
  subseq. converges

Then
$$d(x_n, x) \leq d(x_n, x_{n_k}) + d(x_{n_k}, x) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence,
$\forall \varepsilon > 0 \quad \exists \tilde{K} = \max(N, K) \quad \forall n \geq \tilde{K} \quad d(x_n, x) < \varepsilon \Rightarrow$
$\Rightarrow \lim\limits_{n\to\infty} x_n = x$. ◀

(6) Consider the metric spase c_0 consisting of all sequences $x=(\xi_k)_{k=1}^{\infty}$ which converge to 0. A metric on c_0 is defined as

$$d(x,y)=\max_{k\geq 1}|\xi_k-\eta_k|,\quad x=(\xi_k)_{k=1}^{\infty},\ y=(\eta_k)_{k=1}^{\infty}\in c_0.$$

Prove that c_0 is complete.

► We know that (c,d) is a complete metric space and $c_0\subset c$. If c_0 is closed in c, then (c_0,d) is a complete metric subspace.

We need to show that c_0 is closed in c.

Take $\forall x_n\in c_0$, $n\geq 1$, $x_n\to x$ in c and prove that $x\in c_0$.

Since $x_n\to x$, then

$$\forall\varepsilon>0\ \exists N\geq 1\ \forall n\geq N\quad d(x,x_n)<\frac{\varepsilon}{2}. \qquad (*)$$

We know that $x_n\in c_0 \Leftrightarrow \forall\varepsilon>0\ \exists K\geq 1\ \forall k\geq K$

$$|\xi_k^N|<\frac{\varepsilon}{2},\quad x_N=(\xi_k^N)_{k=1}^{\infty}. \qquad (**)$$

Set $x=(\xi_k)_{k=1}^{\infty}$. Then

$$|\xi_k|=|\xi_k-\xi_k^N+\xi_k^N|\leq|\xi_k-\xi_k^N|+|\xi_k^N|\leq$$

$$\leq\underbrace{\sup_{k\geq 1}|\xi_k-\xi_k^N|}_{=d(x,x_N)}+|\xi_k^N|=\underbrace{d(x,x_N)}_{\text{by }(*)\ <\frac{\varepsilon}{2}}+\underbrace{|\xi_k^N|}_{\text{by }(**)\ <\frac{\varepsilon}{2}}<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.$$

We obtained that

$$\forall k\geq K\quad |\xi_k|<\varepsilon\ \Rightarrow\ x=(\xi_k)_{k=1}^{\infty}\in c_0.$$

Hence, c_0 is closed in c. By Th. 13.7 (c_0,d) is a complete metric space. ▲

7) Show that the set of all real number $\mathbb{R}$ with the metric $d(x,y) = |\arctan x - \arctan y|$, $x, y \in \mathbb{R}$, is not a complete metric space.

▶

Show that $(\mathbb{R}, d)$ is not complete metric space.
Take a Cauchy seq. $\{x_n\}_{n=1}^{\infty}$ and prove that
$\nexists x \in \mathbb{R}$ s.t. $x_n \to x$, $n \to \infty$.

Let $x_n = n$ and check that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy seq.

$$d(x_n, x_m) = |\underbrace{\arctan n}_{\xrightarrow[n\to\infty]{} \frac{\pi}{2}} - \underbrace{\arctan m}_{\xrightarrow[m\to\infty]{} \frac{\pi}{2}}| \to 0, \quad n, m \to \infty.$$

$\Rightarrow$ x_n is a Cauchy seq. in $(\mathbb{R}, d)$.
Show that $\nexists x \in \mathbb{R}$ s.t. $d(x_n, x) \to 0$, $n \to \infty$.
$d(x_n, x) = |\arctan x_n - \arctan x| \xrightarrow[n\to\infty]{} 0$ if $\arctan x = \frac{\pi}{2}$,
but $\nexists x \in \mathbb{R}$: $\arctan x = \frac{\pi}{2}$.

Hence, $(\mathbb{R}, d)$ is not complete. ◀

(8) We define a map $T: c \to \mathbb{R}$ as follows

[4p] $Tx = \lim_{k\to\infty} \xi_k$, $x = (\xi_k)_{k=1}^{\infty} \in c$.

Is the map T continuous? Justify your answer.

▶ We need to show that T is continuous.

T is continuous at x, $x \in c$, if

$\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall y \in c \quad d_c(x,y) < \delta \Rightarrow \delta_{\mathbb{R}}(Tx, Ty) < \varepsilon.$

Take $\varepsilon > 0$ and $x = (\xi_k)_{k=1}^{\infty} \in c$.

Then for $\forall y = (\eta_k)_{k=1}^{\infty} \in c$ we have

$$d_{\mathbb{R}}(Tx, Ty) = |\lim_{k\to\infty} \xi_k - \lim_{k\to\infty} \eta_k| = \lim_{k\to\infty} |\xi_k - \eta_k| \le$$

$$\le \lim_{k\to\infty} \sup_{k\ge 1} |\xi_k - \eta_k| = \sup_{k\ge 1} |\xi_k - \eta_k| = d_c(x,y).$$

$d_{\mathbb{R}}(Tx, Ty) \le d_c(x,y) < \delta = \varepsilon.$

$\exists \delta = \varepsilon : \forall y \in c \quad d_{\mathbb{R}}(Tx, Ty) < \varepsilon$ if $d_c(x,y) < \delta$.

Hence, T is continuous at x $\Rightarrow$ T is continuous. ◀

(9) Consider the metric space $C^1[0;1]$ of all continuously differentiable functions on $[0;1]$. Define the metric on $C^1[0;1]$ as follows

$$d(x,y) = \max_{t\in[0;1]} |x(t)-y(t)| + \max_{t\in[0;1]} |x'(t)-y'(t)|, \quad x,y \in C^1[0;1].$$

Show that $C^1[0;1]$ is a complete metric space.

► We need to show that every Cauchy sequence converges in $C^1[0;1]$.
Take a Cauchy sequence $\{x_n\}_{n=1}^\infty \subset C^1[0;1]$ and prove that

$$\exists x \in C^1[0;1]: \ d_{C^1}(x_n, x) \to 0, \ n\to\infty.$$

• $\{x_n\}_{n=1}^\infty$ is a subseq. in $C[0;1]$.
Show that $\{x_n\}_{n\geq 1}$ is a Cauchy seq. in $C[0;1]$.
Indeed, $\forall\, n,m \geq N$

$$d_C(x_n, x_m) = \max_{t\in[0;1]} |x_n(t)-x_m(t)| \leq \max_{t\in[0;1]} |x_n(t)-x_m(t)| +$$
$$+ \max_{t\in[0;1]} |x_n'(t)-x_m'(t)| = d_{C^1}(x_n, x_m) \to 0, \quad n,m\to\infty$$

$\{x_n\}$ is a Cauchy seq. in $C^1[0;1]$

Then $\exists x \in C[0;1]$ s.t. $x_n \to x$ in $C[0;1]$, that is,

$$\max_{t\in[0;1]} |x_n(t)-x(t)| \to 0, \ n\to\infty.$$

• Consider a seq. $\{x_n'\}_{n=1}^\infty \subset C[0;1]$, where x_n' is a derivative of x_n.
$\{x_n'\}_{n\geq 1}$ is a Cauchy seq. in $C[0;1]$, because

$$d_C(x_n', x_m') = \max_{t\in[0;1]} |x_n'(t)-x_m'(t)| \leq d_{C^1}(x_n, x_m) \to 0, \ n,m\to\infty.$$

Then $\exists\, \tilde{x} \in C[0;1]$ st. $x_n' \to \tilde{x}$ in $C[0;1]$.

• Next
1) check that $x \in C^1[0;1]$, that is, prove that $\exists x'$ which is continuous on $[0;1]$ and $x' = \tilde{x}$;
2) show that $\{x_n\}_{n\geq 1} \subset C^1[0;1]$ converges to x.

1) We know that

$$\int_0^t x_n'(s)ds = x_n(t) - x_n(0) \Rightarrow$$

$$\Rightarrow x_n(t) = x_n(0) + \int_0^t x_n'(s)ds. \qquad (*)$$

Then

$$\lim_{n\to\infty} x_n(t) = \lim_{n\to\infty} x_n(0) + \lim_{n\to\infty} \int_0^t x_n'(s)ds.$$

By dominated conv. theor. all $|x_n'(s)| \le C$ $\forall n \ge 1$, $\forall s \in [0;1]$, because x_n' is bounded in $C[0;1]$. then

$$\lim_{n\to\infty} \int_0^t x_n'(s)ds = \int_0^t \lim_{n\to\infty} x_n'(s)\, ds = \int_0^t \tilde{x}(s)\, ds.$$

From $(*)$

$$\lim_{n\to\infty} x_n(t) = \lim_{n\to\infty} x_n(0) + \lim_{n\to\infty} \int_0^t x_n'(s)\, ds$$

$$\Rightarrow \quad x(t) = \lim_{n\to\infty} x_n(0) + \int_0^t \tilde{x}(s)\, ds$$

and

$$x'(t) = 0 + \tilde{x}(t) \quad \Rightarrow \quad x'(t) = \tilde{x}(t) \quad \forall t \in [0;1].$$

2) Prove that $x_n \to x$ in $C^1[0;1]$.
For this show that $d_{C^1}(x_n, x) \to 0$, $n \to \infty$.

$$d_{C^1}(x_n, x) = \underbrace{\max_{t\in[0;1]} |x_n(t) - x(t)|}_{\to 0} + \underbrace{\max_{t\in[0;1]} |x_n'(t) + \overset{\text{,,}\tilde{x}(t)}{x'(t)}|}_{\to 0} \to 0, n\to\infty.$$

$\Rightarrow x_n \to x$ in $C^1[0;1]$.

Hence, $C^1[0;1]$ is a complete metric space. ▲