Let X denote a normed space with norm $\|\cdot\|$.

① Prove that $\|x\|_p = \left(\sum_{k=1}^{n} |\xi_k|^p\right)^{1/p}$, $x = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$, is not a norm in $\mathbb{R}^n$ for $p<1$ and $n \geq 2$.

▶ Take $0<p<1$, $x = (1, 0, \dots, 0) \in \mathbb{R}^n$, $y = (0, 1, 0, \dots, 0) \in \mathbb{R}^n$ and check the property (N4) of the definition of a norm

$$\|x+y\| \leq \|x\| + \|y\|, \quad \forall x, y \in \mathbb{R}.$$

$$\|x+y\| = \left(\sum_{k=1}^{n} |\xi_k + \eta_k|^p\right)^{1/p} = \left(|1+0|^p + |0+1|^p + |0+0| + \dots + |0+0|^p\right)^{1/p} =$$

$$= 2^{1/p}.$$

$$\|x\| = \left(\sum_{k=1}^{n} |\xi_k|^p\right)^{1/p} = \left(1^p + 0^p + \dots + 0^p\right)^{1/p} = 1,$$

$$\|y\| = \left(\sum_{k=1}^{n} |\eta_k|^p\right)^{1/p} = 1.$$

For $0<p<1$ we have $\frac{1}{p} > 1$ and $\underbrace{2^{1/p}}_{=\|x+y\|} > \underbrace{2^1}_{=\|x\|+\|y\|} \Rightarrow$

$\Rightarrow \|x+y\| > \|x\| + \|y\| \Rightarrow$

$\Rightarrow \|\cdot\|$ does not satist. the property (N4) $\Rightarrow$

$\Rightarrow \|\cdot\|$ is not a norm in $\mathbb{R}^n$ for $0<p<1$ and $n \geq 2$.

For $p \leq 0$ $\|\cdot\|$ is not defined at $x=0$.

Hence, $\|\cdot\|$ is not a norm in $\mathbb{R}^n$ for $p<1$ and $n \geq 2$. ◀

(2) Show that the closed unit ball

$$B_r(x_0) = \{x \in X : \|x - x_0\| \le r\}$$

in X is convex[1] for any $x_0 \in X$ and $r > 0$.

[1] A subset A of a vector space V is said to be convex if for every $x, y \in A$ it implies that

$$\alpha x + (1-\alpha) y \in A.$$

for all $\alpha \in [0;1]$.

▶ Take $\forall x, y \in B_r(x_0)$ and show that for $\forall \alpha \in [0;1]$

$$\alpha x + (1-\alpha) y \in B_r(x_0),$$

that is, $\|\alpha x + (1-\alpha) y - x_0\| \le r$.

$x_0 = \alpha x_0 + (1-\alpha) x_0$. Then

$$\|\alpha x + (1-\alpha) y - \alpha x_0 - (1-\alpha) x_0\| = \|\alpha(x - x_0) + (1-\alpha)(y - x_0)\| \overset{(N4)}{\le}$$

$$\le \|\alpha(x - x_0)\| + \|(1-\alpha)(y - x_0)\| \overset{(N3)}{=} \underbrace{|\alpha|}_{\ge 0} \cdot \|x - x_0\| + \underbrace{|1-\alpha|}_{\ge 0} \cdot \|y - x_0\| =$$

$$= \alpha \cdot \underbrace{\|x - x_0\|}_{\le r, \text{ because } x \in B_r(x_0)} + (1-\alpha) \cdot \underbrace{\|y - x_0\|}_{\le r, \ y \in B_r(x_0)} \le \alpha \cdot r + (1-\alpha) \cdot r = \alpha r + r - \alpha r = r$$

$\Rightarrow \alpha x + (1-\alpha) y \in B_r(x_0) \quad \forall x_0 \in X, r > 0, \forall x, y \in B_r(x_0).$

Hence, $B_r(x_0)$ is convex in X for any $x_0 \in X$ and $r > 0$. ◀

③ Show that the convergences $x_n \to x$, $y_n \to y$ in X and $\alpha_n \to \alpha$ in the field K imply that

$$x_n + y_n \to x + y \quad \text{and} \quad \alpha_n x_n \to \alpha x \quad \text{in } X.$$

▶ $x_n \to x \Rightarrow \|x_n - x\| \to 0,\ n\to\infty$

$y_n \to y \Rightarrow \|y_n - x\| \to 0,\ n\to\infty$ in X.

$\alpha_n \to \alpha \Rightarrow |\alpha_n - \alpha| \to 0,\ n\to\infty$ in K.

- We need to show that $x_n + y_n \to x+y$ in X, that is,

$$\|(x_n+y_n) - (x+y)\| \to 0,\quad n\to\infty.$$

$$\|(x_n+y_n)-(x+y)\| = \|(x_n - x) + (y_n - y)\| \overset{(N4)}{\le} \underbrace{\|x_n - x\|}_{\to 0} + \underbrace{\|y_n - y\|}_{\to 0} \to 0,\ n\to\infty$$

$\Rightarrow x_n + y_n \to x+y$.

- Show that $\alpha_n x_n \to \alpha x$ in X, that is,

$$\|\alpha_n x_n - \alpha x\| \to 0,\quad n\to\infty.$$

$$\|\alpha_n x_n - \alpha x\| = \|\alpha_n x_n - \alpha_n x + \alpha_n x - \alpha x\| \overset{(N4)}{\le} \|\alpha_n x_n - \alpha_n x\| + \|\alpha_n x - \alpha x\|$$

$$= \underbrace{|\alpha_n|}_{\text{bounded, because } \alpha_n \text{ converges}} \cdot \underbrace{\|x_n - x\|}_{\to 0} + \underbrace{|\alpha_n - \alpha|}_{\to 0} \cdot \underbrace{\|x\|}_{\text{constant}} \to 0,\ n\to\infty$$

$\Rightarrow \alpha_n x_n \to \alpha x$ in X. ◀

④ Show that the closure $\bar{Y}$ of a subspace Y of X is again a vector subspace.

▸ $\bar{Y} \subset X$ is called a vector subspace of X if $\forall\, x, y \in \bar{Y}$, $\forall \alpha \in K$

1) $x+y \in \bar{Y}$; 2) $\alpha x \in \bar{Y}$.

Recall that $x \in \bar{Y}$ iff $\exists\, x_n \in Y$ s.t. $x_n \to x$, $n \to \infty$.

1) Take $x, y \in \bar{Y}$ and show that $x+y \in \bar{Y}$.

$x+y \in \bar{Y}$ if $\exists$ a sequence in Y which converges to $x+y$.

Since $x, y \in \bar{Y}$, then $\exists\, x_n \in Y$ and $y_n \in Y$ s.t.

$$x_n \to x, \quad y_n \to y, \quad n \to \infty.$$

$x_n + y_n \in Y$, because Y is a vector subspace and

$x_n + y_n \to x + y$ (from exercise 3).

We obtained that $\exists$ a sequence $x_n + y_n \in Y$ which converges to $x+y \Rightarrow x+y \in \bar{Y}$.

2) Take $x \in \bar{Y}$ and $\forall \alpha \in K$.

Since $x \in \bar{Y}$, then $\exists\, x_n \in Y$ s.t. $x_n \to x$, $n \to \infty$.

Then $\alpha x_n \to \alpha x$ (from exerc. 3).

$\alpha x_n \in Y$, because Y is a vector subspace.

$\Rightarrow \alpha x \in \bar{Y}$.

Hence, $\bar{Y}$ is a vector subspace. ▲

(5) Show that X must be complete, if absolute convergence of any series always implies convergence of that series in X.

▶ Let $\{x_n\}_{n\geq 1}$ be a Cauchy sequence in X.
We have to show that $\{x_n\}_{n\geq 1}$ converges in X.

Since $\{x_n\}_{n\geq 1}$ is a Cauchy sequence,

- for $\varepsilon_1 = \frac{1}{2}$ $\exists N_1$ $\forall n, m \geq N_1$
$$\|x_n - x_m\| < \frac{1}{2}.$$
- for $\varepsilon_2 = \frac{1}{2^2}$ $\exists N:$ $\forall n, m \geq N$
$$\|x_n - x_m\| < \frac{1}{2^2}.$$

Take $N_2 := \max\{N_1+1, N\}$. Then $N_2 > N_1$ and $\forall n, m \geq N$
$$\|x_n - x_m\| < \frac{1}{2^2}.$$

- By induction, for every $k \geq 1$ we can similarly choose $N_k > N_{k-1}$ s.t. $\forall n, m \geq N_k$
$$\|x_n - x_m\| < \frac{1}{2^k}.$$

So, we have constructed a subsequence $N_1 < N_2 < N_3 < \ldots$ such that $\forall k \geq 1$
$$\|x_n - x_m\| < \frac{1}{2^k} \quad \forall n, m \geq N_k.$$

In particular,
$$\|x_{N_{k+1}} - x_{N_k}\| < \frac{1}{2^k} \quad \forall k \geq 1.$$

We write
$$x_{N_k} = x_{N_k} - x_{N_{k-1}} + x_{N_{k-1}} - x_{N_{k-2}} + \ldots + x_{N_2} - x_{N_1} + x_{N_1} - \underbrace{x_{N_0}}_{=0} =$$
$$= \sum_{k=1}^{k} (x_{N_i} - x_{N_{i-1}}).$$

Since $\sum_{i=1}^{\infty} \|x_{N_i} - x_{N_{i-1}}\| \leq \sum_{i=1}^{\infty} \frac{1}{2^{i-1}} = 2$, the series $\sum_{i=1}^{\infty} (x_{N_i} - x_{N_{i-1}})$ converges in X, that is, there exists $x \in X$ such that
$$x_{N_k} = \sum_{k=1}^{k} (x_{N_i} - x_{N_{i-1}}) \longrightarrow x.$$

We have shown that $\{x_n\}_{n\geq 1}$ has a convergent subsequence. Using Exercise 5 HW7, we can conclude that $\{x_n\}_{n\geq 1}$ also convergent in X.

▲

⑥ Show that in a Banach space, an absolutely convergent series is convergent.
[3p.]

▶ We have that $\sum_{n=1}^{\infty} \|x_n\|$ converges and need to show that $\sum_{n=1}^{\infty} x_n$ converges.

Recall that $\sum_{n=1}^{\infty} x_n$ converges iff $S_n = \sum_{k=1}^{n} x_n$, $n \geq 1$, converges.

Prove that $\{S_n\}_{n\geq 1}$ is a Cauchy sequence in X.

set $n \geq m$

$$\|S_n - S_m\| = \left\| \sum_{k=1}^{n} x_k - \sum_{k=1}^{m} x_k \right\| = \left\| \sum_{k=m+1}^{n} x_k \right\| \overset{(N4)}{\leq} \sum_{k=m+1}^{n} \|x_k\| \leq$$

$$\leq \sum_{k=m+1}^{\infty} \|x_k\| \to 0, \text{ because } \sum_{n=1}^{\infty} \|x_n\| \text{ converges}$$

$\Rightarrow \{S_n\}_{n\geq 1}$ is a Cauchy seq.

Since X is a Banach space, then every Cauchy seq. converges $\Rightarrow \{S_n\}_{n\geq 1}$ converges.

Hence, $\sum_{n=1}^{\infty} x_n$ converges. ◀

(7) Let $(Y, \|\cdot\|_Y)$ and $(Z, \|\cdot\|_Z)$ be normed space. Show that the product vector space $X = Y \times Z$ becomes a normed space if we define

$$\|x\| = \max\{\|y\|_Y, \|z\|_Z\}, \quad x = (y, z) \in X.$$

Check also that a sequence $x_n = (y_n, z_n)$, $n \geq 1$, converges to $x = (y, z)$ in X if and only if $y_n \to y$ in Y and $z_n \to z$ in Z.

1. We need to show that $\|\cdot\|$ is a norm in X.
For this let us check the propeties (N1)-(N4).

(N1) $\|x\| \geq 0 \quad \forall\, x \in X$

$\|x\| = \max\{\|y\|_Y, \|z\|_Z\} \geq 0$, because $\|y\|_Y \geq 0$, $\|z\|_Z \geq 0$.

(N2) $\|x\| = 0 \Leftrightarrow x = 0$, that is, $x = (y, z) = (0, 0)$.

$\|x\| = \max\{\|y\|_Y, \|z\|_Z\} = 0 \Leftrightarrow \|y\|_Y = 0$ and $\|z\|_Z = 0$

$\Leftrightarrow y = 0$ and $z = 0 \Rightarrow x = (0; 0)$.

(N3) $\|\alpha x\| = |\alpha| \cdot \|x\|$, $\forall x \in X$, $\forall \alpha \in K$

$\alpha x = (\alpha y, \alpha z)$

$\|\alpha x\| = \max\{\|\alpha y\|_Y, \|\alpha z\|_Z\} = \max\{|\alpha| \cdot \|y\|_Y, |\alpha| \cdot \|z\|_Z\} =$

$= |\alpha| \cdot \max\{\|y\|_Y, \|z\|_Z\} = |\alpha| \cdot \|x\|.$

(N4) $\|x_1 + x_2\| \leq \|x_1\| + \|x_2\|$, $\forall x_1, x_2 \in X$

$x_1 = (y_1, z_1)$, $x_2 = (y_2, z_2)$

$\|x_1 + x_2\| = \max\{\|y_1 + y_2\|_Y; \|z_1 + z_2\|_Z\}.$

If $\|y_1 + y_2\|_Y \geq \|z_1 + z_2\|_Z$, then $\max\{\|y_1 + y_2\|_Y, \|z_1 + z_2\|_Z\} = \|y_1 + y_2\|_Y$.

$\|x_1 + x_2\| = \|y_1 + y_2\|_Y \underset{\|\cdot\|_Y \text{ - norm in } Y}{\leq} \|y_1\|_Y + \|y_2\|_Y \leq$

$\leq \max\{\|y_1\|_Y; \|z_1\|_Z\} + \max\{\|y_2\|_Y; \|z_2\|_Z\} = \|x_1\| + \|x_2\|.$

Similary, if $\|z_1 + z_2\|_Z \geq \|y_1 + y_2\|_Y$, then

$\|x_1 + x_2\| \leq \|x_1\| + \|x_2\|.$

Hence, $\|\cdot\|$ is a norm in X

2. $x_n = (y_n, z_n) \to x = (y, z)$ in X iff $y_n \xrightarrow[\text{in } Y]{} y$, $z_n \to z$ in Z

show that $\|x_n - x\| \to 0 \iff \|y_n - y\|_Y \to 0$, $\|z_n - z\|_Z \to 0$, $n \to \infty$

$\Rightarrow$) Let $x_n \to x$ in X, then $\|x_n - x\| \to 0$, $n \to \infty$, in X.

$\|x_n - x\| = \max\{\|y_n - y\|_Y ; \|z_n - z\|_Z\}$ and we know that

$\underbrace{\|x_n - x\|}_{\to 0} \geq \|y_n - y\|_Y \geq 0$ and $\underbrace{\|x_n - x\|}_{\to 0} \geq \|z_n - z\|_Z \geq 0$

$\Rightarrow \|y_n - y\|_Y \to 0$, $\|z_n - z\|_Z \to 0$ $\Rightarrow$ $z_n \to z$ in Z,
$y_n \to y$ in Y.

$\Leftarrow$) Let $y_n \to y$ in Y, $z_n \to z$ in Z.
show that $x_n \to x$ in X.

$\|x_n - x\| = \max\{\underbrace{\|y_n - y\|_Y}_{\to 0}, \underbrace{\|z_n - z\|_Z}_{\to 0}\} \to 0$ in X,

because $(t_1, t_2) \mapsto \max\{t_1, t_2\}$ is contineeous.

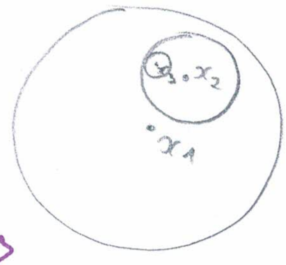
$\Rightarrow x_n \to x$ in X. $\blacktriangle$

(8) Let X be a Banach space and B_n be a family of closed balls in X such that $B_{n+1} \subset B_n$, $n \geq 1$.
Show that

(a) there exists $x \in X$ such that $\bigcap_{n=1}^{\infty} B_n = \{x\}$, if radii r_n of the balls B_n converges to zero.

▶ Set $B_n = B_{r_n}(x_n)$. Consider a sequence $\{x_n\}_{n\geq 1}$, where x_n are centers of balls B_n.
We need to show that:

1) $\{x_n\}_{n\geq 1}$ is a Cauchy seq.;
2) $\exists\, x$ s.t. $x_n \to x$, $n \to \infty$, and $x \in \bigcap_{n=1}^{\infty} B_n$;
3) $\exists!\, x \in \bigcap_{n=1}^{\infty} B_n$ and $\bigcap_{n=1}^{\infty} B_n = \{x\}$.



1) Since $r_n \to 0$, $n \to \infty$, then
$\forall \varepsilon > 0 \;\; \exists N \geq 1 \;\; \forall n \geq N \;\; \|r_n\| < \frac{\varepsilon}{2}$.
Then $\forall k, m \geq N$ $\; x_k \in B_N$ and $x_m \in B_N \Rightarrow$
$\Rightarrow \|x_k - x_N\| \leq r_N < \frac{\varepsilon}{2}$ and $\|x_m - x_N\| \leq r_N < \frac{\varepsilon}{2}$.

$$\|x_k - x_m\| = \|x_k - x_N + x_N - x_m\| \leq \|x_k - x_N\| + \|x_N - x_m\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \quad \forall k, m \geq N.$$

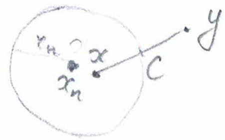
$\Rightarrow \{x_n\}_{n\geq 1}$ is a Cauchy seq.

2) Since X is a Banach space, then every Cauchy seq. converges, that is,
$$\exists x \in X \text{ s.t. } x_n \to x \text{ in } X.$$
Show that $x \in \bigcap_{n=1}^{\infty} B_n$.
Take any n_0, then for all $n \geq n_0$ $\; x_n \in B_{n_0}$.
Since $x_n \to x$ and B_{n_0} is closed, then
$$x \in B_{n_0} \quad \forall n_0 \geq 1$$
$$\Rightarrow x \in \bigcap_{n=1}^{\infty} B_n.$$

3) Let us suppose that $y \neq x$ and show that
$$y \notin \bigcap_{n=1}^{\infty} B_n.$$

If $y \neq x$, then $\|y - x\| = c$ and

$\exists n : r_n < \frac{c}{2} \Rightarrow y \notin B_{r_n}(x_n)$.



Indeed, $c = \|y - x\| \leq \|y - x_n\| + \|x_n - x\| \leq$

$$\leq \|y - x_n\| + r_n < \|y - x_n\| + \frac{c}{2} \Rightarrow$$

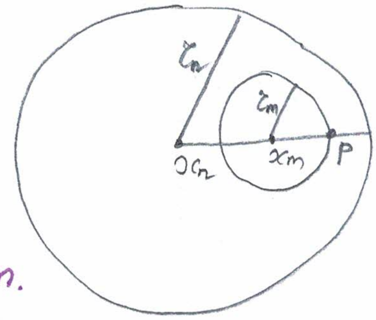
$$\Rightarrow \|y - x_n\| > c - \frac{c}{2} = \frac{c}{2} > r_n \Rightarrow y \notin B_{r_n}(x_n) \Rightarrow$$

$$\Rightarrow y \notin \bigcap_{n=1}^{\infty} B_{r_n}(x_n) = \bigcap_{n=1}^{\infty} B_n .$$

Hence, $\exists!\, x \in X$ s.t. $\bigcap_{n=1}^{\infty} B_n = \{x\}$. ◀

(8) (b) $\bigcap_{n=1}^{\infty} B_n \neq \emptyset$ without the assumption that $r_n \to 0$.

▷ We want to show that $\|x_n - x_m\| \le r_n - r_m$, $n < m$.
The inequality is trivial if $x_n = x_m$.
So, assume that $x_n \neq x_m$.



Set
$$p = x_m + \frac{x_m - x_n}{\|x_m - x_n\|} \cdot r_m.$$

Remark that $\|p - x_m\| = \left\| \frac{x_m - x_n}{\|x_m - x_n\|} r_m \right\| = r_m$.

So, $p \in B_m \subset B_n$.

Then
$$r_n \geq \|p - x_n\| = \left\|x_m + \frac{x_m - x_n}{\|x_m - x_n\|} r_m - x_n\right\| =$$
$$= \left\| (x_m - x_n)\left(1 + \frac{r_m}{\|x_m - x_n\|}\right) \right\| = \|x_m - x_n\| \left(1 + \frac{r_m}{\|x_m - x_n\|}\right) =$$
$$= \|x_m - x_n\| + r_m.$$

Hence, $\|x_m - x_n\| \le r_n - r_m$.

Next, since $\{r_n\}_{n\geq 1}$ and decreases, there exists $r \geq 0$ s.t. $r_n \to r$.
Consequently, for $m > n$
$$\|x_m - x_n\| \le r_n - r_m \to r - r = 0, \quad m, n \to \infty, \ m > n.$$

Consequently, $\{x_n\}_{n\geq 1}$ is a Cauchy sequense and hence, convergent to some x.
Similarly as in part a) $x \in \bigcap_{n=1}^{\infty} B_n$.

◀