Problem sheet 1

1. Show that
$$\int_x^{+\infty} e^{-\frac{y^2}{2}} dy \sim \frac{1}{x} e^{-\frac{x^2}{2}}, \quad x \to +\infty.$$

2. Let $(a_n)_{n \geq 1}$ and $(b_n)_{n \geq 1}$ be two sequences of positive real numbers. We say that they are **logarithmically equivalent** and write $a_n \simeq b_n$ if
$$\lim_{n \to \infty} \frac{1}{n} (\ln a_n - \ln b_n) = 0.$$
   (a) Show that $a_n \simeq b_n$ iff $b_n = a_n e^{o(n)}$.
   (b) Show that $a_n \sim b_n$ implies $a_n \simeq b_n$ and that the inverse implication is not correct.
   (c) Show that $a_n + b_n \simeq \max\{a_n, b_n\}$.

3. Let $\xi_1, \xi_2, \ldots$ be independent Bernoulli distributed random variables with parameter $p = \frac{1}{2}$. Let also $S_n = \xi_1 + \cdots + \xi_n$. Using Theorem 1.1 from the lecture notes, show that
$$\sum_{n=1}^{\infty} \mathbb{P} \left\{ \left| \frac{S_n}{n} - \frac{1}{2} \right| \geq \varepsilon \right\} < \infty,$$
for all $\varepsilon > 0$. Conclude that $\frac{S_n}{n} \to \frac{1}{2}$ a.s. as $n \to \infty$ *(strong low of large numbers)*.
(*Hint:* Use the Borel-Cantelly lemma to show the convergence with probability 1)

4. Prove Theorem 1.1