Problem sheet 10

1. Let $H(\nu|\mu)$ be a relative entropy of ν given μ, where $\nu, \mu \in \mathcal{P}(U)$ and U be a finite space.

   (i) Show that the function $H(\cdot|\mu) : \mathcal{P}(U) \to \mathbb{R}$ is continuous.

   (ii) Prove that $H(\nu|\mu) > 0$ for every $\nu \neq \mu$ and $H(\mu|\mu) = 0$.

   (iii) Show that the function $H(\cdot|\mu)$ is good, that is, the level sets $\{\nu \in \mathcal{P}(U) : \ H(\nu|\mu) \leq \alpha\}$, $\alpha \geq 0$, are compact in $\mathcal{P}(U)$.

2. Let $\xi_1, \xi_2, \ldots$ be independent Bernoulli distributed random variables with parameter $p \in (0,1)$. Using Sanov's theorem and the contraction principle show that the family $(\frac{1}{n} S_n)_{n \geq 1}$ satisfies the large deviation principle with good rate function

$$I(x) = \begin{cases} x \ln \frac{x}{p} + (1-x) \ln \frac{1-x}{1-p} & \text{if } x \in [0,1], \\ +\infty & \text{otherwise,} \end{cases}$$

   where $S_n = \xi_1 + \cdots + \xi_n$.

3. Let f be a continuous and bounded above function from a metric space E to $\mathbb{R}$. Show that for every $n \geq 1$ there exists a family of closed subsets B_k, $k \in [m]$, of E such that $f \leq -n$ on $B_0 := \left(\bigcup_{k=1}^m B_k\right)^c$ and the oscillation of f on each B_k is at most $\frac{1}{n}$.

   Hint: Consider the sets $f^{-1}\left(\left[\frac{k-1}{n}, \frac{k}{n}\right]\right)$, $k \in \mathbb{Z}$.

4. Let A be a subset of E and $f, g : A \to \mathbb{R}$ with $\inf\limits_{x \in A} g(x) > -\infty$. Prove that

$$\inf_{x \in A} f(x) - \inf_{x \in A} g(x) \leq \sup_{x \in A} (f(x) - g(x)).$$