Problem sheet 11

Let E be a complete separable metric space.

1. Let I^K be defined by

$$I^K(x) = I(x) - K(x) - \inf_{y \in E} (I(y) - K(y)), \quad x \in E,$$

for a good rate function I and $K \in \mathrm{C}_b(E)$.

   (i) Show that the function I^K is good.

   (ii) Prove the equality

$$I^K(x) = \sup_{f \in \mathrm{C}_b(E)} \left(f(x) - \Lambda_f^K \right), \quad x \in E,$$

   where $\Lambda_f^K = \sup_{x \in E} \left(f(x) - I^K(x) \right)$.

2. Let F be a closed subset of E. Show that the closed δ-neighborhood $F^\delta = \{x \in E : d(x, F) \le \delta\}$ of F is a closed set and $\bigcap_{\delta > 0} F^\delta = F$.

3. Let $I : E \to [0, \infty]$ be good, F be a closed set and F^δ be the closed δ-neighborhood of F. Show that $\inf_{x \in F^\delta} I(x) \to \inf_{x \in F} I(x)$ as $\delta \to 0$.