# Problem sheet 2

1. Assume that a random variable $\xi$ has a finite first moment $\mathbb{E}\,\xi = \mu$ and let $\varphi$ be the comulant generating function associated with $\xi$. Show that for every $x > \mu$ and all $\lambda < 0$
$$\lambda x - \varphi(\lambda) \leq 0.$$
(*Hint:* Use Jensen's inequality.)

2. Let $\varphi$ be a comulant generating function associated with $\xi$. Show that the function $\varphi$ is differentiable in the interior of the domain $\mathcal{D}_\varphi := \{x \in \mathbb{R} : \ \varphi(x) < \infty\}$. In particular, show that $\varphi'(0) = \mathbb{E}\,\xi$ if $0 \in \mathcal{D}_\varphi^\circ$.
(*Hint:* To show the differentiability of $\varphi$, it is enough to show that $\mathbb{E}\,e^{\lambda\xi}$, $\lambda \in \mathbb{R}$, is differentiable. For the differentiability of the latter function, use the definition of the limit, the dominated convergence theorem and the fact that the function $\frac{e^{\varepsilon a}-1}{\varepsilon} = \int_0^a e^{\varepsilon x} dx$ increases in $\varepsilon > 0$ for each $a \geq 0$.)

3. Show that the Fenchel-Legendre transform of a convex function $f$ is also convex.

4. Show that the Fenchel-Legendre transform of the comulant generating function associated with $N(0,1)$ coincides with $\frac{x^2}{2}$.

5. Suppose that $\varphi^*$ is the Fenchel-Legendre transform of the cumulant generating function of a random variable $\xi$ with $\mathbb{E}\,\xi = \mu$. Show that
   (i) $\varphi^*(x) \geq 0$ for all $x \in \mathbb{R}$. (*Hint:* Use the fact that $\varphi(0) = 0$)
   (ii) $\varphi^*(\mu) = 0$. (*Hint:* Use (i) and Jensen's inequality to show that $\varphi^*(\mu) \leq 0$)
   (iii) $\varphi^*$ increases on $[\mu, \infty)$ and decreases on $(-\infty, \mu]$. (*Hint:* Use the convexity of $\varphi^*$)

6. Let $\varphi^*$ be the Fenchel-Legendre transform of the comulant generating function of a random variable $\xi$. Let also $\beta = \operatorname{ess\,sup} \xi < \infty$. Show that $\varphi^*(x) = +\infty$ for all $x > \beta$.
*Hint:* Show that $\lim_{\lambda\to+\infty}(\lambda x - \varphi(\lambda)) = +\infty$.