Problem sheet 3

1. Let $\xi_1, \xi_2, \dots$ be independent identically distributed random variables. Consider a non-negative Borel measurable function $f : \mathbb{R} \to [0, \infty)$ such that $\mathbb{E}\, f(\xi_1) \in (0, \infty)$. Define the family of independent random variables $\eta_1, \eta_2, \dots$ with distribution

$$\mathbb{P}\left\{\eta_i \in B\right\} = \frac{1}{C}\mathbb{E}\left[f(\xi_i)\mathbb{I}_{\{\xi_i \in B\}}\right], \quad B \in \mathcal{B}(\mathbb{R}),$$

where $C = \mathbb{E}\, f(\xi_i)$ is the normalizing constant.

   (a) Find the distribution of η_1, if ξ_1 has the exponential distribution with parameter $\lambda > 0$, and $f(x) = e^{-\alpha x}$, $x \in \mathbb{R}$, where $\alpha > -\lambda$ is a positive constants.

   (b) Show that for every $n \in \mathbb{N}$ and $B_i \in \mathcal{B}(\mathbb{R})$

$$\mathbb{P}\left\{\eta_1 \in B_1, \dots, \eta_n \in B_n\right\} = \frac{1}{C^n}\mathbb{E}\left[f(\xi_1)\dots f(\xi_n)\mathbb{I}_{\{\xi_1 \in B_1, \dots, \xi_n \in B_n\}}\right].$$

   (c) Show that

$$\mathbb{E}\, g(\eta_1, \dots, \eta_n) = \frac{1}{C^n}\mathbb{E}\left[f(\xi_1)\dots f(\xi_n)g(\xi_1, \dots, \xi_n)\right],$$

   for any Borel measurable function $g : \mathbb{R}^n \to \mathbb{R}$.

2. Let a_n, b_n, $n \geq 1$, be positive real numbers. Show that

$$\varlimsup_{n\to\infty} \frac{1}{n}\ln(a_n + b_n) = \varlimsup_{n\to\infty} \frac{1}{n}\ln a_n \vee \varlimsup_{n\to\infty} \frac{1}{n}\ln b_n,$$

where $a \vee b$ denotes the maximum of the set $\{a, b\}$.

3. Let $\eta_1, \eta_2 \sim N(0, 1)$. Let also for every $\varepsilon > 0$ a random variable ξ_ε have the distribution defined as follows

$$\mathbb{P}\left\{\xi_\varepsilon \in A\right\} = \frac{1}{2}\mathbb{P}\left\{-1 + \sqrt{\varepsilon}\eta_1 \in A\right\} + \frac{1}{2}\mathbb{P}\left\{1 + \sqrt{\varepsilon}\eta_2 \in A\right\}$$

for all Borel sets A. Show that the family $(\xi_\varepsilon)_{\varepsilon>0}$ satisfies the LDP with rate function

$$I(x) = \frac{1}{2}\min\left\{(x-1)^2, (x+1)^2\right\}, \quad x \in \mathbb{R}.$$