Problem sheet 4

1. Let $a_n > b_n$, $n \geq 1$, be positive real numbers such that there exist limits (probably infinite)

$$a := \lim_{n\to\infty} \frac{1}{n} \ln a_n \quad \text{and} \quad b := \lim_{n\to\infty} \frac{1}{n} \ln b_n$$

and $a > b$. Show that

$$\lim_{n\to\infty} \frac{1}{n} \ln(a_n - b_n) = a.$$

2. Let $(\xi_\varepsilon)_{\varepsilon>0}$ satisfies the LDP in E with rate function I. Show that

   a) if A is such that $\inf_{x\in A^\circ} I(x) = \inf_{x\in \bar{A}} I(x)$, then

$$\lim_{\varepsilon\to 0} \varepsilon \ln \mathbb{P}\left\{\xi_\varepsilon \in A\right\} = -\inf_{x\in A} I(x);$$

   b) $\inf_{x\in E} I(x) = 0$.

3. Let $E = \mathbb{R}$ and $\xi \sim N(0,1)$. Show that the family $(\varepsilon\xi)_{\varepsilon>0}$ satisfies the LDP with rate function

$$I(x) = \begin{cases} +\infty & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

4. For any random vector $\xi \in \mathbb{R}^d$ and non-singular $d \times d$ matrix A, show that $\varphi_{A\xi}(\lambda) = \varphi_\xi(\lambda A)$ and $\varphi^*_{A\xi}(x) = \varphi^*_\xi(A^{-1}x)$.

5. For any pair of independent random vectors ξ and η show that $\varphi_{\xi,\eta}(\lambda,\mu) = \varphi_\xi(\lambda) + \varphi_\eta(\mu)$ and $\varphi^*_{\xi,\eta}(x,y) = \varphi^*_\xi(x) + \varphi^*_\eta(y)$.

6. Let $\xi_1, \xi_2, \ldots$ be independent normal distributed random vectors in $\mathbb{R}^d$ with mean 0 and positively defined covariance matrix C. Show that the empirical means $\left(\frac{1}{n}S_n\right)_{n\geq 1}$ satisfies the LDP in $\mathbb{R}^d$ and find the corresponding rate function I.