# Problem sheet 5

1. Let $f_\lambda : E \to \mathbb{R}$, $\lambda \in \mathbb{R}$, be a family of continuous functions, where $E$ is a metric space. Show that the function $f(x) = \sup_{\lambda \in \mathbb{R}} f_\lambda(x)$, $x \in E$, is lower semi-continuous.

2. Let $E$ be a metric space and $f : E \to [-\infty, +\infty]$. Define

$$f_{\mathrm{lsc}}(x) = \sup \left\{ \inf_{y \in G} f(y) : \ G \ni x \text{ and } G \text{ is open} \right\}. \tag{1}$$

   (a) Show that if $x_n \to x$, then $f_{\mathrm{lsc}}(x) \le \underline{\lim}_{n\to\infty} f(x_n)$.

   (*Hint:* Use Lemma 5.2, namely that the function $f_{\mathrm{lsc}}$ is lower semi-continuous and $f_{\mathrm{lsc}} \le f$)

   (b) Show that for each the supremum in (1) can only be taken over all ball with center $x$, namely

$$f_{\mathrm{lsc}}(x) = \sup_{r>0} \inf_{y \in B_r(x)} f(y) \tag{2}$$

   (*Hint:* Use the fact that any open set $G$ containing $x$ also contains a ball $B_r(x)$ for some $r > 0$. It will allow to prove the inequality $f_{\mathrm{lsc}}(x) \le \sup_{r>0} \inf_{y \in B_r(x)} f(y)$. The inverse inequality just follows from the observation that supremum in the right hand side of (2) is taken over smaller family of open sets)

   (c) Prove that for each $x \in E$ there is a sequence $x_n \to x$ such that $f(x_n) \to f_{\mathrm{lsc}}(x)$ (the constant sequence $x_n = x$ is allowed here). This gives the alternate definition

$$f_{\mathrm{lsc}}(x) = \min \left\{ f(x), \underline{\lim}_{y \to x} f(y) \right\}.$$

   (*Hint:* Use part b) of the exercise to construct the corresponding sequence $x_n$, $n \ge 1$)

3. Let $(E, d)$ be a metric space and $f : E \to [0, +\infty)$ be lower semi-continuous. Define for each $n \in \mathbb{N}$ the function

$$f_n(x) = \inf_{y \in E} \{ f(y) + n \cdot d(x, y) \}, \quad x \in E.$$

   Show that

   (a) $f_n$ increases, that is, $f_n(x) \le f_{n+1}(x)$ for all $x \in E$ and $n \in \mathbb{N}$;

   (b) $f_n$ is continuous for each $n \in \mathbb{N}$;

   (c) $f(x) = \lim_{n\to\infty} f_n(x) = \sup_{n \in \mathbb{N}} f_n(x)$ for all $x \in E$.