Problem sheet 6

1. Let E be a complete and separable metric space.

   a) Show that exponential tightness implies tightness for a countable family of random variables.

   (*Hint:* Prove a similar inequality to one in the previous exercise and then use the fact that any random element on a complete and separable metric space is tight

   b) Show that tightness does not imply exponential tightness.

2. Let $(\xi_\varepsilon)_{\varepsilon>0}$ be a family of random variables in $\mathbb{R}$ such that there exist $\lambda > 0$ and $\kappa > 0$ such that $\mathbb{E}\, e^{\frac{\lambda}{\varepsilon}|\xi_\varepsilon|} \leq \kappa^{\frac{1}{\varepsilon}}$ for all $\varepsilon > 0$. Show that this family is exponentially tight.

   (*Hint:* Use Chebyshev's inequality)

3. Find a simpler proof of Proposition 6.3 in the case $E = \mathbb{R}^d$.

   (*Hint:* Cover a level set $\{x \in R^d : \ I(x) \leq \beta\}$ by an open ball and use the upper bound)