Problem sheet 7

1. Let $h \in \mathrm{C}^1[0,T]$ and $w(t)$, $t \in [0,T]$, be a Brownian motion. Show that

$$\int_0^T h(t)dw(t) = h(T)w(T) - h(0)w(0) - \int_0^T h'(t)w(t)dt.$$

(*Hint:* Take a partition $0 = t_0 < t_1 < \cdots < t_n = T$ and check first that functions $h_n = \sum_{k=1}^n h(t_k)\mathbb{I}_{[t_{k-1},t_k)}$ converge to h in $L_2[0,T]$ as the mesh of partition goes to 0, using e.g. the uniform continuity of h on $[0,T]$. Next show that

$$\sum_{k=1}^n h(t_{k-1})(w(t_k) - w(t_{k-1})) = h(t_{n-1})w(T) - h(0)w(0) - \sum_{k=1}^{n-1} w(t_k)(h(t_k) - h(t_{k-1}))$$

Then prove that the first partial sum converges to the integral $\int_0^T h(t)dw(t)$ in L_2 and the second partial sum converges to $\int_0^T w(t)dh(t)$ a.s. as the mesh of partition goes to 0)

2. Let $N(t)$, $t \geq 0$, be a Poisson process. Define $N_n(t) = \frac{1}{n}N(nt)$, $t \geq 0$, for all $n \geq 1$.

   (a) Show that for every $t > 0$ the family $(N_n(t))_{n\geq 1}$ satisfies the LDP in $\mathbb{R}$ (with $a_n = \frac{1}{n}$) and find the corresponding rate function.

   (b) Show that for every $t_1 < t_2 < \cdots < t_d$ the family $((N_n(t_1), \ldots, N_n(t_d)))_{n\geq 1}$ satisfies the LDP in $\mathbb{R}^d$ (with $a_n = \frac{1}{n}$) and find the corresponding rate function.

   (c) Which form should have the rate function in the LDP for the family of processes $\{N_n(t),\ t \in [0,T]\}_{n\geq 1}$ in the space $\mathrm{C}_0[0,T]$?

3. Show that for any $f \in H_0^2[0,T]$ there exists a sequence $(f_n)_{n\geq 1}$ from $\mathrm{C}_0^2[0,T]$ such that $f_n \to f$ in $\mathrm{C}_0[0,T]$ and $I(f_n) \to I(f)$ as $n \to \infty$, where

$$I(f) = \begin{cases} \frac{1}{2}\int_0^T \dot{f}^2(t)dt & \text{if } f \in H_0^2[0,T], \\ +\infty & \text{otherwise.} \end{cases}$$

(*Hint:* Use first the fact that $\mathrm{C}^1[0,T]$ is dense in $L_2[0,T]$. Then show that if $h_n \to h$ in $L_2[0,T]$, then $\int_0^\cdot h_n(s)ds$ tends to $\int_0^\cdot h(s)ds$ in $\mathrm{C}_0[0,T]$, using Hölder's inequality)