Problem sheet 8

1. Show that for every $a \in \mathbb{R}$ and $\delta > 0$

$$\mathbb{P}\left\{w_{\sigma^2}(t) + at < \delta, \quad t \in [0,T]\right\} > 0,$$

where $w_{\sigma^2}(t)$, $t \geq 0$, is a Brownian motion with diffusion rate σ^2.

(*Hint:* Use the Cameron-Marting formula and the fact that $\sup_{t\in[0,T]} w_{\sigma^2}(t)$ and $|w_{\sigma^2}(T)|$ have the same distribution[1])

2. Let $g \in L_2[0,T]$ and for every $h \in \mathrm{C}^1[0,T]$

$$h(T)f(T) - \int_0^T h'(t)f(t)dt = \int_0^T h(t)g(t)dt.$$

Show that f is absolutely continuous with $\dot{f} = g$.

(*Hint:* Consider the function $\tilde{f}(t) = \int_0^t g(s)ds$ and apply to $\int_0^T h(t)g(t)dt$ the integration by parts formula)

3. Let I be a good rate function on E and f be a continuous function from E to S. Show that the infimum in

$$J(y) = \inf\left\{I(x) : \; f(x) = y\right\} = \inf_{f^{-1}(\{y\})} I, \quad y \in S.$$

is attained, that is, there exists $x \in E$ such that $f(x) = y$ and $J(y) = I(x)$.

4. Let $\Phi : \mathrm{C}_0[0,T] \to \mathrm{C}_0[0,T]$ be defined in the proof of Theorem 8.7.

   (a) Show that the function Φ is bijective.

   (b) Prove that $g \in H_0^2[0,T]$ if and only if $f = \Phi(g) \in H_0^2[0,T]$.

   (c) Show that $\dot{g} = \dot{f} - a(f)$ almost everywhere for every $g \in H_0^2[0,T]$ and $f = \Phi(g)$.

[1] see Proposition 13.13 [Kallenberg]