Problem sheet 9

1. Let I be a good rate function on E and f be a continuous function from E to S. Show that the infimum in
$$J(y) = \inf\{I(x): \ f(x) = y\} = \inf_{f^{-1}(\{y\})} I, \quad y \in S.$$
is attained, that is, there exists $x \in E$ such that $f(x) = y$ and $J(y) = I(x)$.

2. Let U be a finite set and $\mathcal{P}(U)$ be the metric space of all probability measures on U equipped with the total variation distance.

   (i) Let $|\nu|_{TV}$ denote the total variation of a signed measure[1] on U. Show that
   $$|\mu - \nu|_{TV} = \sum_{i=1}^{d} |\mu(\{u_i\}) - \nu(\{u_i\})|.$$
   Therefore, the convergence of a sequence $(\nu_n)_{n\geq 1}$ to ν in $\mathcal{P}(U)$ is equivalent to the convergence of $\nu_n(\{u_i\}) \to \nu(\{u_i\})$, $n \to \infty$, for each $i \in [d]$.

   (ii) Show that a sequence $(\nu_n)_{n\geq 1}$ converges in ν in $\mathcal{P}(U)$ if and only it $\nu_n \to \nu$ weakly.

   (iii) Prove that the space $\mathcal{P}(U)$ is complete and separable.
   (*Hint:* Use the isometry between $\mathcal{P}(U)$ and the simplex $\Delta = \{(x_1, \ldots, x_d) \in \mathbb{R}^d : \ x_1 + \cdots + x_d = 1\}$)

3. Let $X_1, X_2, \ldots$ be independent random variables taking values from a finite space U and have distribution μ. Set
$$\mu_n := \frac{1}{n} \sum_{k=1}^{n} \delta_{X_k} \quad n \geq 1.$$
Show that $\mu_n \to \mu$ in $\mathcal{P}(U)$ a.s.
(*Hint:* Use the previous exercise and the strong law of large numbers)

[1]The total variation $|\nu|_{TV}$ of a signed measure ν on U is defined as $|\nu|_{TV} = \sup_{\pi} \sum_{A\in\pi} |\nu(A)|$, where is taken over all partitions π of the set U